Constructive Matrix Theory for Higher Order Interaction (towards the double scaling limit)

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T. Krajewski and V. Rivasseau, VS, arXiv:1712.05670;
VS, work in progress

Motivation

Matrices are everywhere in physics, randomness is also everywhere.
Random matrices are applicable to:

- the description of the energy spectra of heavy nuclei
- the large N_c limit of QCD
- random surfaces, 2d quantum gravity
- transport in disordered systems
- string theory
- number theory
- biology
- ....

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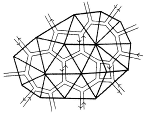
The sum over the random geometries



- The partition function is

$$Z = \log \int dM\, e^{-S(M)}$$

with hermitian $N \times N$ matrices M.

- The partition function can be expand as

$$Z(g) = N^2 Z_0(g) + Z_1(g) + N^{-2} Z_2(g) + \ldots = \sum_h N^{2-2h} Z_h(g)\,.$$

- All coefficients $Z_h(g)$ diverge at the same critical value of the coupling $g = g_c$.
- In the limit $g \to g_c$ the contributions from higher genus h are enhanced.
- We take the limits $N \to \infty$ and $g \to g_c$ simultaneously.
- This results in a coherent contribution from all genus surfaces.

Motivation

Many questions concerning the $1/N$ expansion of matrix models remain unanswered, the main are:

- $1/N$ expansion is a result of the reshuffling of the divergent perturbation theory.
- Matrix models are defined for a fixed (positive) sign of the coupling constant, but $g_c < 0$.

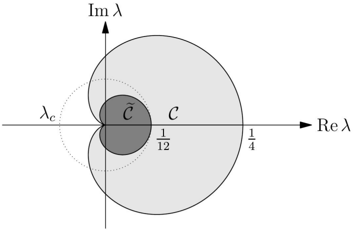
The key to resolve these problems is to study the analytic properties of matrix models!

What is known

The domain of analyticity for the free energy of the quartic matrix model

$$F = \log \int dM e^{-\mathrm{Tr} M^2 - \frac{\lambda}{\sqrt{2N}} \mathrm{Tr} M^4}$$

is proven to be:



[Orthogonal polynomials: P. Bleher, A. Its 2002];
[LVE: V. Rivasseau, 2007; R. Gurau, T. Krajewski 2014].

Our research

- Can we say something about different interactions?
- Yes!!!
- Is it possible to construct analytic continuation to the region including $\lambda < 0$?
- We will see....

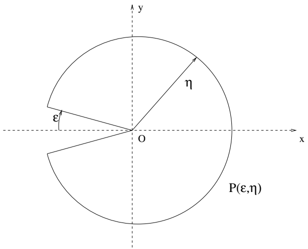
The Model

We study the models with monomial interactions of arbitrarily high even order:

$$Z(\lambda, N) := \int dMdM^\dagger \, e^{-NS(M,M^\dagger)}$$

$$S(M, M^\dagger) := \mathrm{Tr}\{MM^\dagger + \lambda(MM^\dagger)^p\} \, ,$$

where M is a complex matrix $N \times N$, $p \geq 2$ is integer, λ is complex.
The main result: the free energy is analytic for λ in an open domain, $P(\epsilon, \eta) := \{0 < |\lambda| < \eta, |\arg \lambda| < \pi - \epsilon\}$



Loop Vertex Expansion

To proof the main result we apply and develop the LVE machinery, which, in contrast with traditional constructive methods, is *not* based on cluster expansions nor involves small/large field conditions.

- Like the Feynman's perturbative expansion, the LVE allows to compute connected quantities at a glance: log(forests) = *trees*.
- The LVE is an *explicit repacking* of infinitely many subsets of pieces of Feynman amplitudes.
- The convergence of the LVE implies analyticity in the domain *uniform* in N and Borel summability of the usual perturbation series.

Main steps of LVE

1. The divergence of the standard perturbation theory is caused by the too singular growth of the interaction potential at large fields. Therefore, we derive an effective action $S_{eff}(M)$, providing
   polynomial interaction ====> Log-type interaction.
2. Taylor expansion
$$e^{S_{eff}(M)} = \sum_{n=0}^{\infty} \frac{(S_{eff}(M))^n}{n!} .$$
3. Replication of fields, by introducing a degenerate Gaussian measure, so
$$(S_{eff}(M))^n \quad ====> \quad \prod_{i}^{n} S_{eff}(M_i) .$$
4. Application of the BKAR forest formula.
5. Taking the log by reducing the sum over forests to the sum over trees.
6. Derivation of the bounds for tree LVE amplitudes.

Effective action (1)

The partition function is

$$\begin{aligned} Z(\lambda, N) &:= \int d\widetilde{M} d\widetilde{M}^\dagger \, e^{-NS(\widetilde{M}, \widetilde{M}^\dagger)}, \\ S(\widetilde{M}, \widetilde{M}^\dagger) &:= \mathrm{Tr}\{\widetilde{M}\widetilde{M}^\dagger + \lambda(\widetilde{M}\widetilde{M}^\dagger)^p\}. \end{aligned}$$

To derive an effective action, we perform a change of variables

$$MM^\dagger = \widetilde{M}\widetilde{M}^\dagger + \lambda(\widetilde{M}\widetilde{M}^\dagger)^p.$$

Then,

$$\widetilde{M}\widetilde{M}^\dagger = MM^\dagger \, T_p(-\lambda(MM^\dagger)^{p-1}),$$

where T_p is a solution of the Fuss-Catalan algebraic equation

$$zT_p^p(z) - T_p(z) + 1 = 0.$$

Effective action (2)

We define

$$X := MM^\dagger\,, \quad A(X) := XT_p(-\lambda X^{p-1})\,.$$

The Jacobian is

$$J = \left|\frac{\delta A(X)}{\delta X}\right| = \left|\frac{A(X)\otimes \mathbf{1} - \mathbf{1}\otimes A(X)}{X\otimes \mathbf{1} - \mathbf{1}\otimes X}\right|$$

and the effective action is

$$S_{eff}(M, M^\dagger) = \log J = \log\left[\mathbf{1}_\otimes + \lambda\sum_{k=0}^{p-1} A^k(X)\otimes A^{p-1-k}(X)\right].$$

Holomorphic calculus

In the following forest/tree expansion we need to compute multiple derivatives ∂_M, $\partial_{M^\dagger}$, therefore we need to simplify the effective action.

Given a holomorphic function f on a domain containing the spectrum of a square matrix X, Cauchy's integral formula yields a convenient expression for $f(X)$,

$$f(X) = \oint_\Gamma dw \frac{f(w)}{w - X},$$

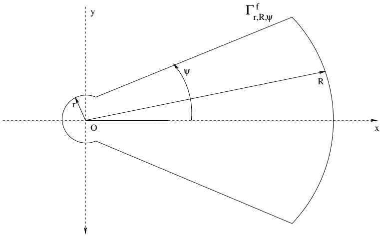
provided the contour Γ encloses the full spectrum of X.

Holomorphic calculus

We can therefore write

$$A(X) = \oint_\Gamma du \; a(\lambda, u) \, \frac{1}{u - X}$$

where $a(\lambda, z) = z T_p(-\lambda z^{p-1})$ and the contour Γ is a *finite* keyhole contour enclosing all the spectrum of X.



The matrix derivative can be easily obtained as

$$\frac{\partial A}{\partial X} = \oint_\Gamma du \; a(\lambda, u) \frac{1}{u - X} \otimes \frac{1}{u - X}.$$

Effective action (3)

The effective action is now given by

$$S_{eff}(\lambda, X) = \int_0^\lambda dt \oint_{\Gamma_1} dv_1 \oint_{\Gamma_2} dv_2 \Big\{ \oint_{\Gamma_0} du\, \phi(t, u, v_1, v_2) \\ \psi(t, v_1, v_2) \Big\} \mathcal{R}(v_1, v_2, X)\,,$$

where $\phi(\lambda, u, v_1, v_2)$ and $\psi(\lambda, v_1, v_2)$ are scalar functions and

$$\mathcal{R}(v_1, v_2, X) = \left[\mathrm{Tr} \frac{1}{v_1 - X} \right] \left[\mathrm{Tr} \frac{1}{v_2 - X} \right].$$

How to compute $\log Z$

The effective action provides a way to generate convergent expansion for the partition function

$$Z(\lambda, N) \;=\; \sum_{n=0}^{\infty} \frac{1}{n!} \int dM dM^\dagger \, \exp\{-N \mathrm{Tr} X\} \, S_{eff}^{n} \, .$$

To compute the logarithm we apply the forest/tree expansion:

forests ====> log ====> trees

Theorem (Brydges-Kennedy-Abdesselam-Rivasseau). *Let $\phi : \mathbb{R}^{\frac{n(n-1)}{2}} \to \mathbb{C}$ be a smooth, sufficiently derivable function. Then:*

$$\phi(1, \ldots, 1) = \sum_{F \text{ forest}} \int_0^1 \prod_{(i,j) \in F} du_{ij} \left(\frac{\partial^{|E(F)|} \phi}{\prod_{(i,j) \in F} \partial x_{ij}} \right) (v_{ij}^F) \, ,$$

where v_{ij}^F is given by:

$$v_{ij}^F = \begin{cases} \inf_{(k,l) \in P_{i \leftrightarrow j}^F} u_{kl} & \text{if } P_{i \leftrightarrow j}^F \text{ exists} \\ 0 & \text{if } P_{i \leftrightarrow j}^F \text{ does not exist} \end{cases} ,$$

and $|E(F)|$ is the number of edges in the forest F.

BKAR forest formula

$n = 2$

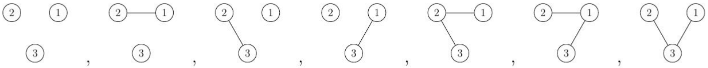
$$\phi(1) = \phi(0) + \int_0^1 dt_{12} \Big(\frac{\partial \phi}{\partial x_{12}} \Big)(t_{12})$$

The first term corresponds to the empty forest ($|E(F)| = 0$) and the second one to the full forest ($|E(F)| = 1$).

BKAR forest formula

$n = 3$

$$
\begin{aligned}
\phi(1,1,1) = \phi(0,0,0) &+ \int_{[0,1]} dt_{12} \left(\frac{\partial \phi}{\partial x_{12}} \right) (t_{12}, 0, 0) \\
&+ \int_{[0,1]} dt_{23} \left(\frac{\partial \phi}{\partial x_{23}} \right) (0, t_{23}, 0) + \int_{[0,1]} dt_{13} \left(\frac{\partial \phi}{\partial x_{13}} \right) (0, 0, t_{13}) \\
&+ \int_{[0,1]^2} dt_{12} dt_{23} \left(\frac{\partial^2 \phi}{\partial x_{12} \partial x_{23}} \right) (t_{12}, t_{23}, \inf(t_{12}, t_{23})) \\
&+ \int_{[0,1]^2} dt_{12} dt_{13} \left(\frac{\partial^2 \phi}{\partial x_{12} \partial x_{13}} \right) (t_{12}, \inf(t_{12}, t_{13}), t_{13}) \\
&+ \int_{[0,1]^2} dt_{23} dt_{13} \left(\frac{\partial^2 \phi}{\partial x_{23} \partial x_{13}} \right) (\inf(t_{23}, t_{13}), t_{23}, t_{13}) \ .
\end{aligned}
$$



Preparing the application of the forest formula

To generate a convergent LVE, we start by expanding $\exp[S_{eff}(X)]$

$$Z(\lambda, N) \;=\; \sum_{n=0}^{\infty} \frac{1}{n!} \int dM dM^\dagger \exp\{-N \mathrm{Tr} X\} \, S_{eff}^n \, .$$

The next step (replicas) is to replace (for the order n) the integral over the single $N \times N$ complex matrix M by an integral over an n-tuple of such $N \times N$ matrices $M_i, 1 \leq i \leq n$.

$$d\mu =====> d\mu_C$$

with a degenerate covariance $C_{ij} = N^{-1} \;\; \forall i, j$.

$$\int d\mu_C M^\dagger_{i|ab} M_{j|cd} = C_{ij} \delta_{ad} \delta_{bc},$$

$M_{i|ab}$ is the matrix element in the row a and column b of the matrix M_i.

$$d\mu_C \Leftrightarrow d\mu \delta(M_1 - M_2) \cdots \delta(M_{n-1} - M_n)$$

Preparing the application of the forest formula

Now the partition function is

$$Z(\lambda, N) = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\mu_C \prod_{i=1}^{n} S_{eff}(M_i)\,,$$

it can be represented as a sum over the set $\mathfrak{F}_n$ of forests $\mathcal{F}$ on n labeled vertices by applying the BKAR formula.

For this, we replace the covariance $C_{ij} = N^{-1}$ by $C_{ij}(x) = N^{-1} x_{ij}$ ($x_{ij} = x_{ji}$) evaluated at $x_{ij} = 1$ for $i \neq j$ and $C_{ii}(x) = N^{-1}\ \forall i$.

The derivative with respect to x_{ij} transforms into derivatives with respect to M_i and $M_j^\dagger$:

$$\partial_{\mathcal{F}} \;\; ====> \;\; \partial_{\mathcal{F}}^{M} = \prod_{(i,j)\in\mathcal{F}} \mathrm{Tr}\left[\frac{\partial}{\partial M_i^\dagger}\frac{\partial}{\partial M_j}\right].$$

$$\partial_M \mathrm{Tr}\frac{1}{v-X} \;\; = \;\; \mathrm{Tr}\left[\frac{1}{v-X}\otimes M^\dagger \frac{1}{v-X}\right]$$

$$\begin{aligned}
\partial_M \partial_{M^\dagger} \mathrm{Tr}\frac{1}{v-X} \;\; &= \;\; \mathrm{Tr}\left[\frac{1}{v-X}M \otimes \frac{1}{v-X} \otimes M^\dagger \frac{1}{v-X}\right] \\
&+ \;\; \mathrm{Tr}\left[\frac{1}{v-X}\otimes M^\dagger \frac{1}{v-X}M \otimes \frac{1}{v-X}\right] \\
&+ \;\; \mathrm{Tr}\left[\frac{1}{v-X}\otimes \mathbf{1} \otimes \frac{1}{v-X}\right].
\end{aligned}$$

The latter derivatives connect "loop vertices".

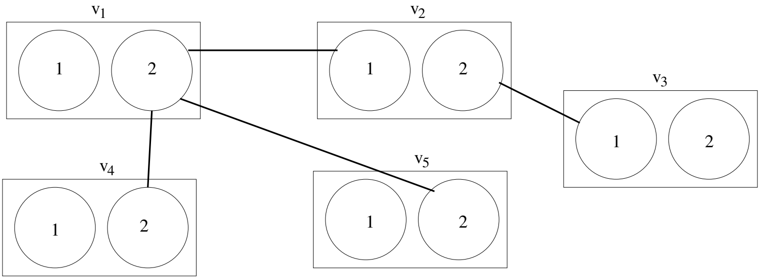


Figure: A tree of $n-1$ lines on n loop vertices (depicted as rectangular boxes, hence here $n=5$) defines a forest of $n+1$ connected components or cycles $\mathcal{C}$ on the $2n$ elementary loops, since each vertex contains exactly two loops. To each such cycle corresponds a trace of a given product of operators in the LVE.

Bounds

- Factorization of the traces provides the possibility to use the trace bound

$$|\mathrm{Tr}[O...O]| \leq \|O\|...\|O\|\,.$$

- On the keyhole contours, the derivatives of the matrix part of the effective action are bounded by

$$\begin{aligned}
\|\frac{1}{v^i_j - X^i}\| &\leq K(1+|v^i_j|)^{-1}, \\
\|\frac{1}{v^i_j - X^i}M^i\| &\leq K(1+|v^i_j|)^{-1/2}\,...
\end{aligned}$$

and for the scalar part we have:

$$\begin{aligned}
|T_p(z)| &\leq \frac{K}{(1+|z|)^{1/p}}, \\
|\frac{d}{dz}T_p(z)| &\leq \frac{K}{(1+|z|)^{1+\frac{1}{p}}}.
\end{aligned}$$

Bounds

- For each tree amplitude, uniformly in N

$$|A_{\mathcal{T}}(\lambda, N)| \leq K^n |\lambda|^{\kappa_p n}$$

- The number of trees grows just as $n!$,
- what is compensated by the symmetry factor $\frac{1}{n!}$.

The main theorem

Theorem

For any $\epsilon > 0$ there exists η small enough such that the LVE expansion is absolutely convergent and defines an analytic function of λ, uniformly bounded in N, in the domain

$$P(\epsilon, \eta) := \{0 < |\lambda| < \eta, |\arg \lambda| < \pi - \epsilon\},$$

a domain which is uniform in N. Here absolutely convergent and uniformly bounded in N means that for fixed ϵ and η as above there exists a constant K independent of N such that for $\lambda \in P(\epsilon, \eta)$

$$\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{T} \in \mathfrak{T}_n} |A_{\mathcal{T}}| \leq K < \infty.$$

Towards the double scaling limit

- The first step is to approach negative couplings
- The problem of the LVE is in the divergence of the logarithmic action at $\lambda < 0$, for instance at $p = 2$, the logarithmic action (obtained by the intermediate field representation) is

$$S_{\log} = \log\left(1 - i\sqrt{\lambda}A\right)$$

  and its derivatives are of the form

$$\frac{1}{1 - i\sqrt{\lambda}A} \otimes .... \otimes \frac{1}{1 - i\sqrt{\lambda}A}$$

- Can we do something with these divergences?

Avoiding divergences

- We can avoid divergences at $\lambda < 0$ if we remove the dependence on λ from the logarithm.
- Consider the partition function

$$Z(\lambda, N) = \int dMdM^\dagger \, e^{-N\mathrm{Tr}\{MM^\dagger + \lambda (MM^\dagger)^p\}}$$

- The first step, what we do is a trivial rescaling of the nominator part (only!) of the normalized matrix integral

$$Z(\lambda, N) = \lambda^{-\frac{N^2}{p}} \int dMdM^\dagger \, e^{-NS(M,M^\dagger)},$$

$$S(M, M^\dagger) = \mathrm{Tr}\{(\lambda^{-1/p} - \alpha^{-1/p})MM^\dagger + \alpha^{-1/p}MM^\dagger + (MM^\dagger)^p\}$$

Avoiding divergences

- Now we can perform the of change variables as

$$\widetilde{M}\widetilde{M}^\dagger = \alpha^{-1/p} M M^\dagger + (M M^\dagger)^p \,,$$

- resulting in the action with two terms

$$\mathcal{S}_A(\widetilde{M}\widetilde{M}^\dagger) = N\,(\lambda^{-1/p} - \alpha^{-1/p})\alpha^{1/p}\mathrm{Tr}[A(\widetilde{M}\widetilde{M}^\dagger)]\,,$$

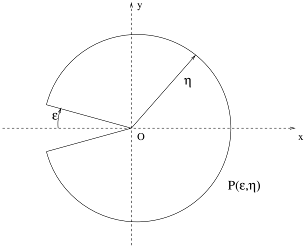
$$\mathcal{S}_{\log}(\widetilde{M}\widetilde{M}^\dagger) = -\mathrm{Tr}_\otimes \log\left[\mathbf{1}_\otimes + \alpha^{-1/p}\sum_{k=0}^{p-1} A^k(\widetilde{M}\widetilde{M}^\dagger)\otimes A^{p-1-k}(\widetilde{M}\widetilde{M}^\dagger)\right]$$

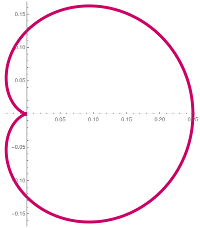
where

$$A(\widetilde{M}\widetilde{M}^\dagger) = \widetilde{M}\widetilde{M}^\dagger\, T_p(-\alpha\widetilde{M}\widetilde{M}^\dagger)\,.$$

α and λ domains

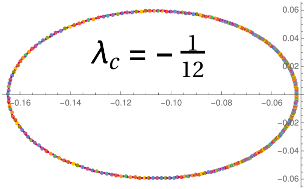
- We require
$$|\lambda^{-1/p} - \alpha^{-1/p}| \leq 1\,,$$
this gives a condition for the λ-domain, depending on α.
- The α-domain is determined by the convergence of the Loop Vertex Expansion

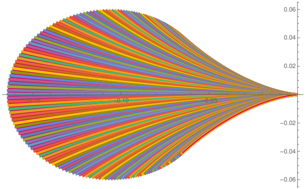




α and λ domains

- When $p = 2$, $\lambda_c = -\frac{1}{12}$.
- Then, the domain in α, which we need to cover $-\frac{1}{12} < \lambda_c \leq 0$, according to $|\lambda^{-1/p} - \alpha^{-1/p}| \leq 1$, is





Conclusions and Outlook

- We conjecture the intersection of the needed and obtained domains in α.
- The results of the first part are also derived for the Hermitian matrix models.
- The techniques used in the work are based on the re-parametrization invariance and highlight its importance.
- The utilization of the holomorphic calculus methods drastically simplifies the construction.
- The latter simplification provides the chance to look from a new perspective to the QFT models, tensor models...