# Emergence of geometry in stochastic systems


## Nobuyuki Matsumoto (Kyoto University)

work in collaboration with M. Fukuma (KU) and N. Umeda (PwC)



arXiv:1705.0609, JHEP 1712 (2017) 001
arXiv:1806.10915 (to appear in JHEP)


**motivation** ∃analogies between quantum systems and stochastic systems [Nelson(1966), Parisi,Wu(1981)]

➡ Can quantum gravity be formulated by using stochastic processes?

**main results** [We here focus on Markov chains as stochastic processes.]

- For a given Markov chain, we introduce a geometry in configuration space by defining the distance between configurations.
- We show that
  - AdS geometry emerges for the simulated tempering algorithm. [Marinari,Parisi(1992)]
  - we can optimize parameters in numerical simulations geometrically. (– practical use of the distance)

## (A) Definition of the distance

- We first prepare some quantities to define the distance:
  - $\mathcal{M} \equiv \{x\}$: configuration space, $S(x)$: action
  - Markov chain with a transition matrix $P(x|y)$ $(\equiv \langle x|\hat{P}|y\rangle)$:

$$x_0 \overset{\hat{P}}{\to} x_1 \overset{\hat{P}}{\to} x_2 \overset{\hat{P}}{\to} \ldots$$

Probability distribution after $n$ steps: $P_n(x|x_0) \equiv \langle x|\hat{P}^n|x_0\rangle$

- Assumptions on $\hat{P}$:
  - ① unique convergence: $P_n(x|x_0) \overset{n\to\infty}{\longrightarrow} \frac{e^{-S(x)}}{Z}$ $(\forall x_0)$ $\left(Z \equiv \int dx\, e^{-S(x)}\right)$
  - ② detailed balance: $P(x|y)e^{-S(y)} = P(y|x)e^{-S(x)}$
  - ③ eigenvalues are all positive

- Transfer matrix: $\hat{T} \equiv e^{S(\hat{x})/2}\hat{P}e^{-S(\hat{x})/2}$

$\hat{T}$ is positive and symmetric ( due to ③ and ② above).

Spectral decomposition of $\hat{T}$:

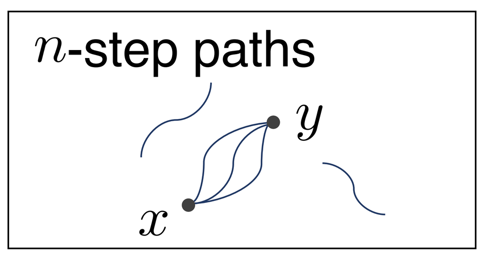
$$\hat{T} = |0\rangle\langle 0| + \sum_{k\geq 1}\lambda_k|k\rangle\langle k| \quad \left(\langle x|0\rangle = \sqrt{\tfrac{e^{-S(x)}}{Z}}\right) \ (1 > \lambda_1 \geq \lambda_2 \geq \cdots)$$

Relaxation to the equilibrium can be understood as the decoupling of higher modes: $\hat{T}^n \overset{n\to\infty}{\longrightarrow} |0\rangle\langle 0|$.

- Connectivity $f_n(x,y)$:

$f_n(x,y) \equiv$ (probability to find a path connecting $x$ and $y$ in the set of $n$-step paths in equilibrium)

$$= P_n(x|y)\cdot\frac{e^{-S(y)}}{Z}$$



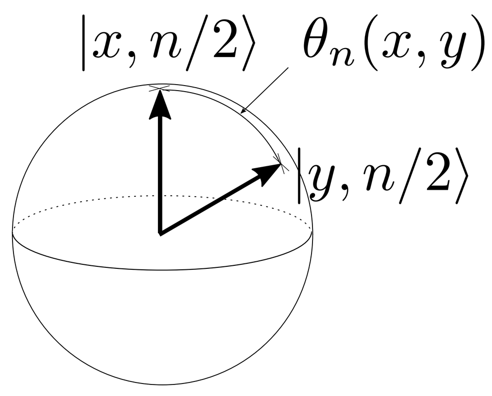

Normalized connectivity $F_n(x,y)$:

$$F_n(x,y) \equiv \frac{f_n(x,y)}{\sqrt{f_n(x,x)f_n(y,y)}} = \frac{\langle x|\hat{T}^n|y\rangle}{\sqrt{\langle x|\hat{T}^n|x\rangle\langle y|\hat{T}^n|y\rangle}}$$

Notes
- ∘ $F_n(x,x) = 1$
- ∘ With normalized states $|x,n/2\rangle \equiv \frac{\hat{T}^{n/2}|x\rangle}{\|\,\hat{T}^{n/2}|x\rangle\,\|}$, $F_n(x,y) = \langle x,n/2|y,n/2\rangle$.

- **We define the distance by**

$$\theta_n(x,y) \equiv \arccos F_n(x,y) \, .$$




$\theta_n(x,y)$ is actually an inner angle between the normalized states $|x,n/2\rangle$, $|y,n/2\rangle$.

**Basic properties**

- $\theta_n(x,y)$ satisfies the axioms of distance.
- If $x$ can be easily reached from $y$, $\theta_n(x,y)$ is small.
- $\theta_n(x,y) \overset{n\to\infty}{\longrightarrow} 0$ $(\forall x,y)$

- Instead of $\theta_n(x,y)$ we can also use:

$$d_n(x,y) \equiv \sqrt{-2\ln F_n(x,y)} \longleftarrow \text{we use this in the following}$$

- Universality of distance

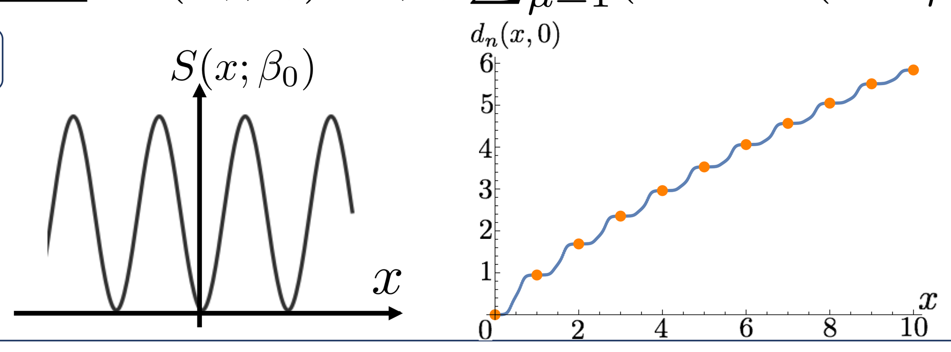
Statement

There are many algorithms to construct $\hat{P}$ satisfying ①-③ above. Among them, for a class of algorithms which generate local moves in configuration space, large scale behavior of $d_n(x,y)$ does not depend on the details of the algorithms.

- Examples
  - quadratic action: $S(x) = \frac{\omega}{2}x^2$

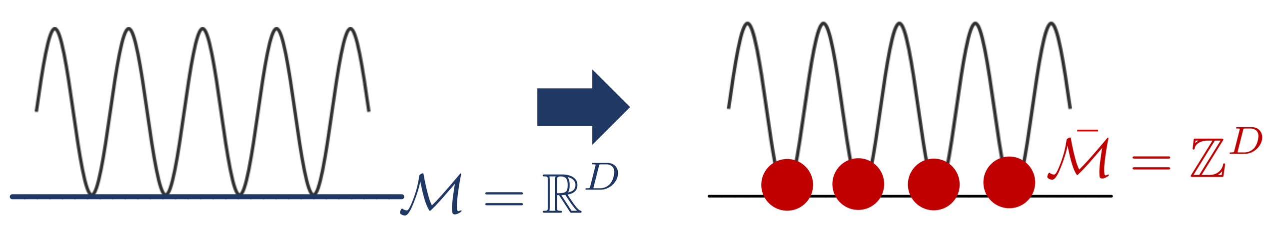
    analytic form of the distance: $d_n(x,y) = \sqrt{\frac{\omega}{2\sinh(\omega t)}}|x-y|$ (using the Langevin algorithm)
  - cosine action: $S(\mathbf{x};\beta_0) = \beta_0\sum_{\mu=1}^{D}(1-\cos(2\pi x_\mu))$ ----- (⋆)

    $(D=1)$




- To investigate the large scale geometry of the configuration space, we coarse-grain the configuration space.
  (identify configurations in the same mode as a single configuration)

e.g. for the cosine action(⋆)




$d_n(x,y)$ does satisfy the triangle inequality after coarse-graining.

- Metric on $\bar{\mathcal{M}}$:

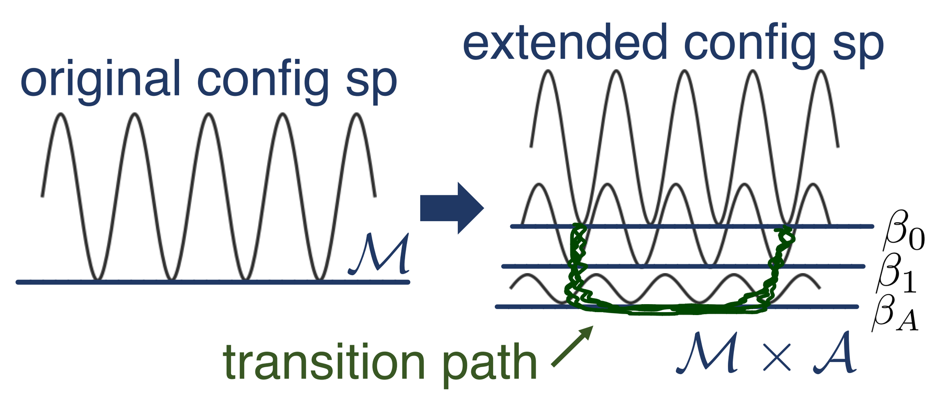
$$ds^2 \equiv d_n^2(x, x+dx) = \sum_{\mu,\nu} g_{\mu\nu}(x)dx_\mu dx_\nu. \quad (x, x+dx : \text{nearby points in } \bar{\mathcal{M}})$$

## (B) Emergence of AdS geometry

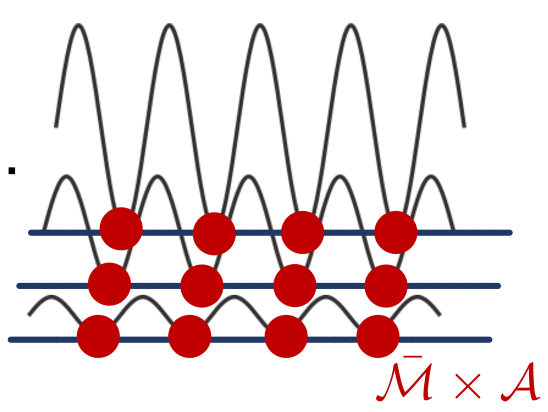
- We consider a system with highly multimodal equilibrium distribution and with highly degenerate vacua. (typical action: the cosine action (⋆))
- We implement the simulated tempering algorithm:
  (an algorithm to accelerate relaxation [Marinari,Parisi(1992)])
  - We extend the configuration space from $\mathcal{M}$ to $\mathcal{M}\times\mathcal{A} \equiv \{x, \beta_a\}$.

    ($\beta_a$: the overall coefficient of the action; $\beta_0(\text{original value}) > \beta_1 > \cdots > \beta_A$)

    ⬇ different modes are now connected in the small $\beta_a$ region



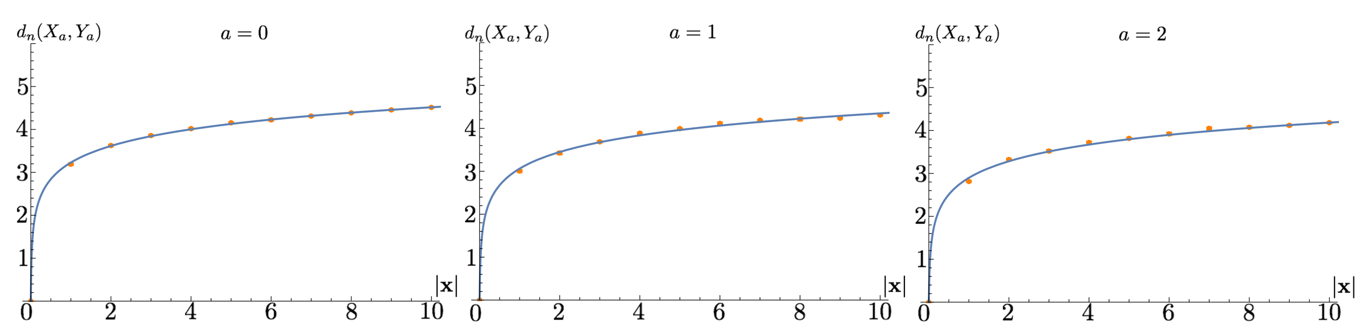

- To investigate the large scale geometry, we introduce extended, coarse-grained configuration space $\bar{\mathcal{M}}\times\mathcal{A}$.




- **One can show that the metric on $\bar{\mathcal{M}}\times\mathcal{A}$ is AdS :**

$$ds^2 = l^2\left(\frac{d\beta^2}{\beta^2} + \alpha\beta^q\sum_{\mu=1}^{D}dx_\mu^2\right).$$

numerical verification




- : numerical result of $d_n(x,y)$
— : analytic geodesic distance with parameters
$l = 0.0404$, $\alpha = 2.34\times 10^5$, $q = 0.289$

## (C) Geometrical optimization

- Knowing geometry, we can optimize parameters in numerical algorithms.

e.g. the simulated tempering

- We optimize the functional form of $\beta_a = \beta(a)$ in such a way that the distances between different modes are minimized.

This should correspond to the case in which the metric in the $\beta$ direction becomes flat when $a$ is used as a coordinate:

$$\frac{d\beta^2}{\beta^2} = \frac{\beta(a)'^2}{\beta(a)^2}da^2 \equiv (\text{const.})da^2.$$

$$\therefore\ \beta_a = \beta_0 e^{-Ra} \quad (R:\text{const.})$$

numerical verification

colored points: metastable states obtained in an optimization process