Canonical tensor model through data analysis —Dimensions, topologies, and geometries—

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Canonical Tensor Model

The canonical tensor model (CTM) describes the time development of the fuzzy space corresponds to a Cauchy surface Σ. The dynamical variables of this model are a real symmetric three-index tensor Q_{abc} and its canonical conjugate P_{abc}, which satisfy Poisson brackets, (σ:permutation)

$$\{Q_{abc}, P_{def}\} = \sum_{\sigma} \delta_{a\sigma_d}\delta_{a\sigma_e}\delta_{a\sigma_f},$$

$$\{Q_{abc}, Q_{def}\} = \{P_{abc}, P_{def}\} = 0,$$

and the indices run from 1 to N. The CTM has two kinds of first-class constraints[6]:

$$\mathcal{H}_a = \frac{1}{2} P_{abc}P_{bde}Q_{cde},$$

$$\mathcal{H}_{ab} = \frac{1}{4}\left(Q_{acd}P_{bcd} - Q_{bcd}P_{acd}\right),$$

and these satisfy the Poisson bracket algebras

$$\{H(N_a), H(M_a)\} = H([\tilde{N}, \tilde{M}]_{ab}),$$
$$\{H(N_{ab}), H(M_a)\} = H(N_{ab}M_b),$$
$$\{H(N_{ab}), H(M_{ab})\} = H([N, M]_{ab}),$$

where $H(N_a) = N_a\mathcal{H}_a$, $H(N_{ab}) = N_{ab}\mathcal{H}_{ab}$ and $\tilde{N}_{ab} = P_{abc}N_c$. These constraints are correspond to the Hamiltonian and momentum constraints in canonical gravity[2]. Quantization of this model can be performed straightforwardly, and the fuzzy space counterpart of the Wheeler-DeWitt equations will be

$$\hat{\mathcal{H}}_a |\Psi\rangle = \hat{\mathcal{H}}_{ab} |\Psi\rangle = 0.$$

P_{abc} : physical meaning

P_{abc} was originally introduced as the structure constant of the algebra of functions f_a on a commutative nonassociative fuzzy space[3]:

$$f_a f_b = P_{ab}{}^c f_c.$$

But in this study, we considered the normal space (i.e. commutative associative space) for simplicity and computability. So if the orthonormal condition is imposed on the basis function $f_a(x)$:

$$\delta_{ab} = \int_{\Sigma} \mathrm{d}x \sqrt{g} f_a(x) f_b(x),$$

then P_{abc} can be obtained by

$$P_{abc} = \int_{\Sigma} \mathrm{d}x \sqrt{g} f_a(x) f_b(x) f_c(x) \quad (1)$$

where $g = \det g_{ij}$ is the determinant of the metric tensor on Σ.

$f_a(x)$: general notes

- Well known theorem:
"If Σ is a compact manifold, then the eigenfunctions of the Laplace-Beltrami operator ∇^2 on Σ form an orthogonal basis for $L^2(\Sigma)$."
Thanks to this, the orthogonalized basis $\{f_a(x)\}$ can be automatically obtained by the Helmholtz equation:

$$(\nabla^2 + m_a^2) f_a = 0$$

if considering manifold Σ is compact.
- In the case of the existence of the boundary $\partial\Sigma \neq \varnothing$, one needs to impose a boundary condition (e.g., Dirichret and Neumann) on $f_a(x)$.

P_{abc} : pragmatic definition

- In general, the index a runs from 1 to $N = \infty$. But to perform the numerical calculation, one have to restrict N to finite. This "sharp cut-off" causes the bad behavior to v_a^i in (2). So it is better to use "smeared" basis $f_a(x)$ to define P_{abc}:

$$\tilde{f}_a(x) = e^{\nabla^2/L^2} f_a(x),$$

with a constant $L \lesssim$(the maximal value of m_a).

points?

- P_{abc} doesn't seem to have any notion of points because of the summation over whole space in (1). Can one extract the information of points from the tensor?
- There is a useful technique for this purpose, known as tensor-rank decomposition. This method represents the tensor by the sum of products of R vectors $\{v_a^i\}$:

$$P_{abc} = \sum_{i=1}^{R} v_a^i v_b^i v_c^i. \quad (2)$$

(2) is similar to (1), so the index i may correspond to the point of the space. This seems to be a correct intuition from the results of numerical calculations.

notes on the tensor-rank decomposition[7]

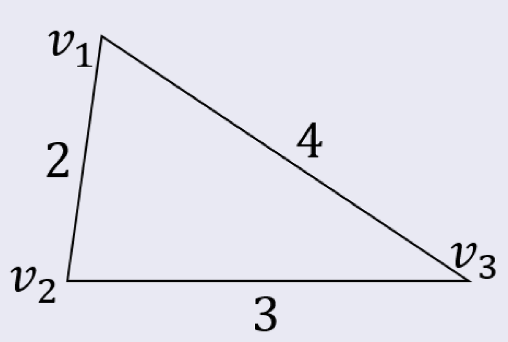
- "The rank of the tensor" is defined as the available minimal value of R of the tensor.
- It is known that the computation of the rank of a given tensor is NP-hard problem[4]. So we gave up to use the true rank, and used the approximation method like

$$P_{abc} = \sum_{i=1}^{R} v_a^i v_b^i v_c^i + \Delta P_{abc} \quad (3)$$

with sufficiently small error $(\Delta P_{abc})^2/(P_{abc})^2$.
- This method causes a new question, for instance, the universality of the results (i.e., can one obtain a similar result from the different R?). From the numerical calculation, the results (like right figures) also seem to be independent of the value of R.

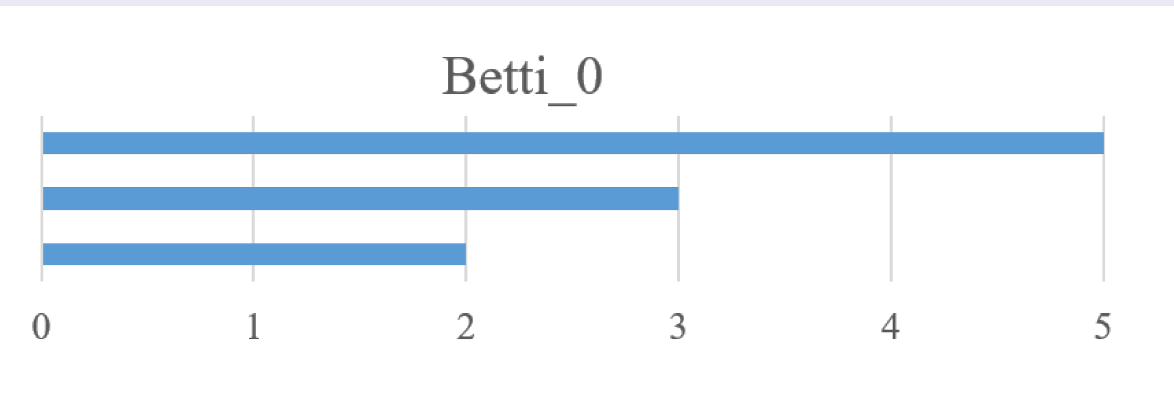
Persistent homology[5]

Prepare the vertex set V and distance $d(v_i, v_j)$ $(v_i, v_j \in V)$.
Vietris-Rips stream $VR(V, u)$ is the mapping from a real parameter u to a simplicial complex that satisfy
(i) $[v] \in VR(V, u)$ for all vertex $v \in V$.
(ii) n-simplex $[v_0 v_1 \ldots v_n] \in VR(V, u)$ iff $d(v_i, v_j) \leq u$ for all edges $[v_i v_j] \in [v_0 v_1 \ldots v_n]$.
simple example
$V = v_1, v_2, v_3,$
$d(v_1, v_2) = 2, d(v_2, v_3) = 3, d(v_3, v_1) = 4$



u	Vietris-Rips complex	Betti number
$0 \leq u < 2$		$B_0 = 3$
$2 \leq u < 3$		$B_0 = 2$
$3 \leq u < 4$		$B_0 = 1$
$4 \leq u$		$B_0 = 1$

Betti interval or barcode



example 1 : 2-sphere S^2

Let the coordinates on S^2 to be (θ, φ), and then the natural basis function is the set of spherical harmonics $\{Y_{l,m}(\theta, \varphi)\}$. The right figure is generated from the condition "two points i and j are connected iff $v_a^i v_a^j > 0.2$."

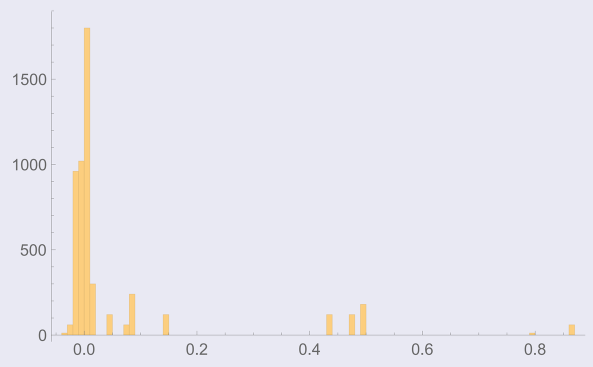


Figure: The histgram of the values of $v_a^i v_a^j$.

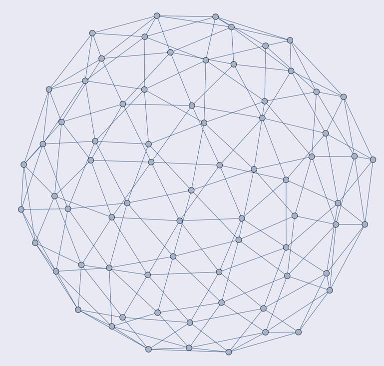
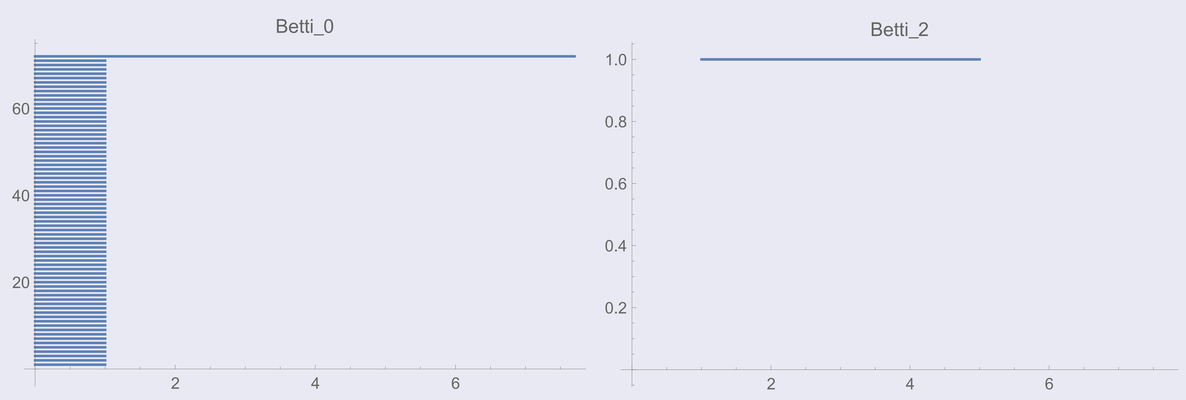


Figure: The resulted S^2 with $N = 36, R = 72$ and $L = 5$.

S^2 has the following Betti numbers for all $\mathbb{Z}_n$:

$$(B_0, B_1, B_2) = (1, 0, 1).$$

The resulted barcodes ($\mathbb{Z}_2$ coefficients) are consistent with this.



example 2 : Klein bottle K^2

The list of the basis function on K^2 is omitted, but its derivation is not so difficult. The right figure is generated from the condition $v_a^i v_a^j > 0.05$.
Ref. Betti numbers of K^2 are

$$(B_0, B_1, B_2) = (1, 2, 1)$$

for $\mathbb{Z}_2$ coefficients, and

$$(B_0, B_1, B_2) = (1, 1, 0)$$

for $\mathbb{Z}_3$ coefficients.

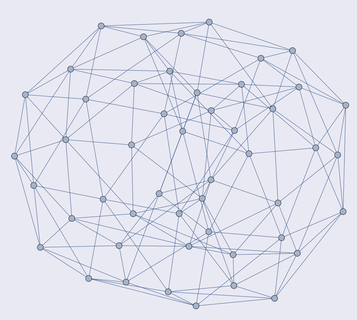
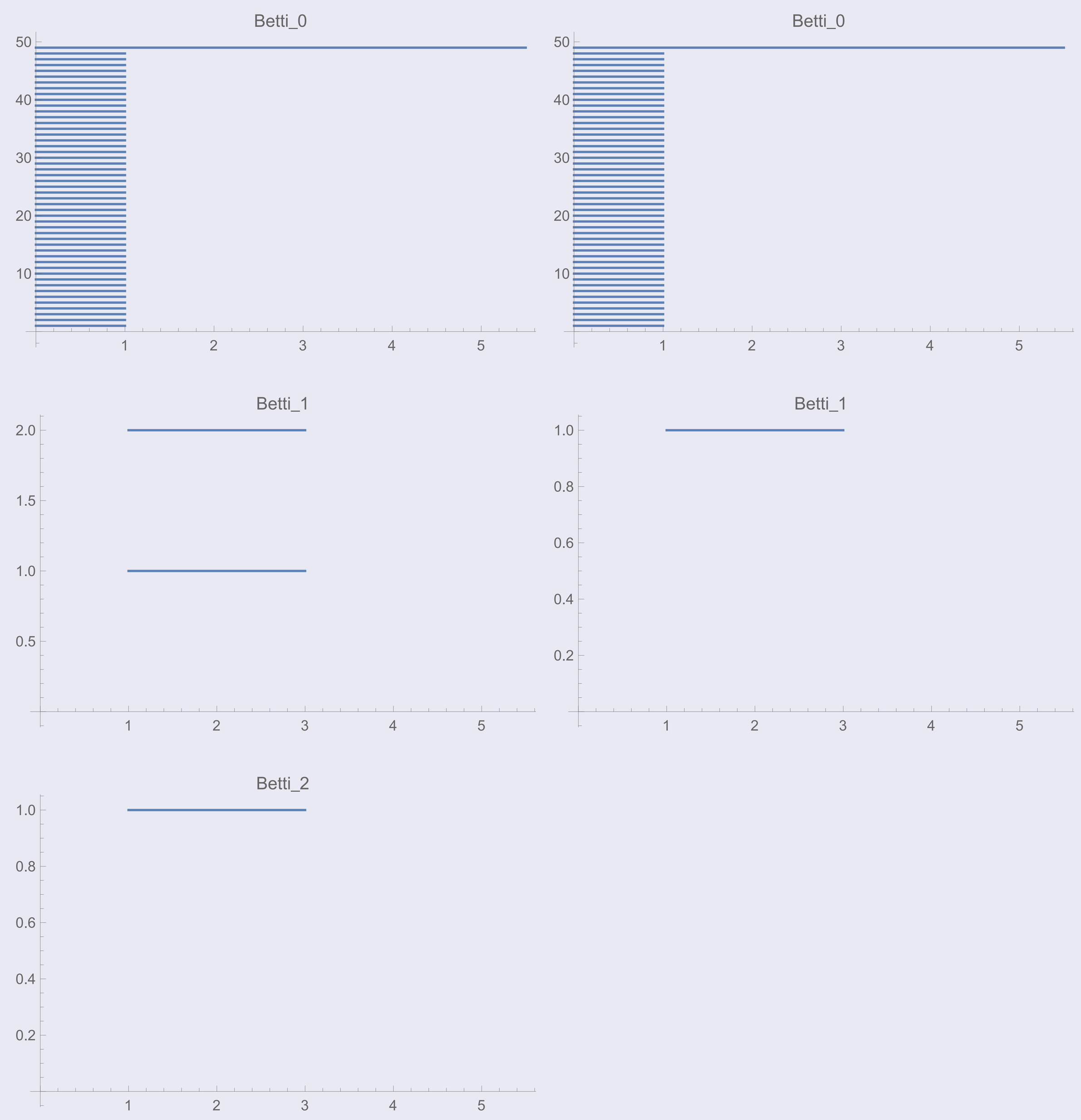


Figure: The resulted K^2 with $N = R = 49$ and $L = 3$.



barcodes. left:$\mathbb{Z}_2$ coefficients, right:$\mathbb{Z}_3$ coefficients.

References

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[2]N. Sasakura, Y. Sato (2015)
[3]N. Sasakura (2011)
[4]C. J. Hillar, L. Lim (2013)
[5]G. Carlsson (2009)
[6]N. Sasakura (2012)
[7]P. Comon (2014) etc.