On the maximal size of subsets which only a few "distances" appear of vector spaces over finite fields

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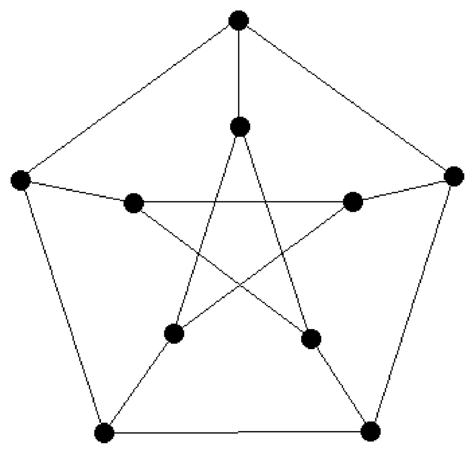
Terms in graph theory

Let G be a k-regular graph (every vertex connects to k edges).

- $V(G)$: vertex set of G, $E(G)$: edge set of G
- $A(G) = (a_{ij})$: adjacency matrix of G with $V(G) = \{v_1, \ldots, v_n\}$.

$$a_{ij} = \begin{cases} \#(\text{edges between } v_i \text{ and } v_j) & \text{if } i \neq j \\ \#(\text{loops connecting to } v_i) & \text{if } i = j \end{cases}$$



- $\mathrm{Spec}(G) = \{\theta \mid \theta: \text{eigenvalue of } A(G)\}$.

It is known that $\forall\theta \in \mathrm{Spec}(G)$, $\theta \in [-k, k]$

- $\boldsymbol{\lambda(G) := \max\{|\theta| \mid \theta \in \mathrm{Spec}(G) \text{ s.t. } |\theta| \neq k\}}$.

Erdős distance problem over $\mathbb{R}^d$

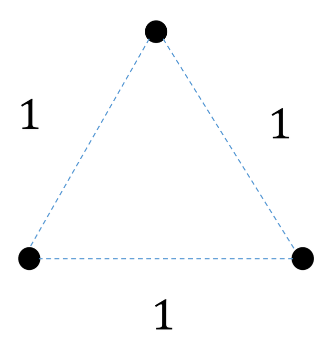
Let $X \subset \mathbb{R}^d$ be a finite set.

- $e(x, y) = \sqrt{(x-y)^t \cdot (x-y)}$: Euclidean distance
- $\Delta(X) = \{e(x, y) \mid x, y \in X, x \neq y\}$.

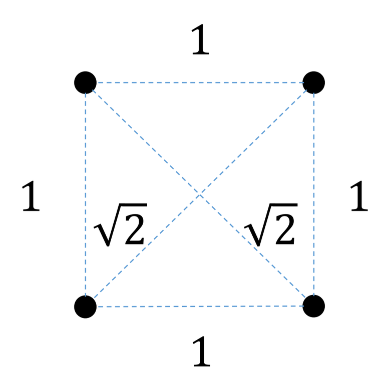
Problem (Erdős, 1946) For each $n \in \mathbb{N}$,

$$g_d(n) := \min\{|\Delta(X)| \mid X \subset \mathbb{R}^d \text{ s.t. } |X| = n\} = ?$$

e.g.



$g_2(3) = 1$



$g_2(4) = 2$

(Guth-N.H.Katz, 2015)

In the case of $d = 2$, $g_2(n) \geq \Omega(n^{1-o(1)})$ for $n \gg 0$.

(Solymosi-Vu, 2008)

In the case of $d \geq 3$, $g_d(n) \geq \Omega\left(n^{\frac{2}{d} - \frac{2}{d(d+2)}}\right)$ for $n \gg 0$.

Erdős distance problem over $\mathbb{F}_q^d$

Let $X \subset \mathbb{F}_q^d$.

- $Q(x) = x^t \cdot x = \sum_{i=1}^d x_i^2$
- $\Delta_Q(X) = \{Q(x-y) \mid x, y \in X, x \neq y\}$.

Problem (Iosevich-Rudnev 2008) For each $n \in \mathbb{N}$,

$$\min\{|\Delta_Q(X)| \mid X \subset \mathbb{F}_q^d \text{ s.t. } |X| = n\} = ?$$

Theorem (Iosevich-Rudnev 2008, Vinh 2008)

If $|X| \geq Cq^{\frac{d}{2}}$ for sufficiently large constant C,

$$|\Delta_Q(X)| \geq \min\left\{q, \frac{|X|}{q^{\frac{d-1}{2}}}\right\}.$$

s-distance sets over $\mathbb{R}^d$

A ***s*-distance set** over $\mathbb{R}^d$ is a finite set $X \subset \mathbb{R}^d$ s.t. $|\Delta(X)| = s$.

Problem (Erdős, 1946).

$$D_d(s) = \max\{|X| \mid X: s\text{-distance set over } \mathbb{R}^d\} = ?$$

Theorem (Bannai-Bannai-Stanton, 1983 et.al.).

$$D_d(s) \leq \binom{d+s}{s}.$$

There seem no corresponding results for $\mathbb{F}_q^d$!

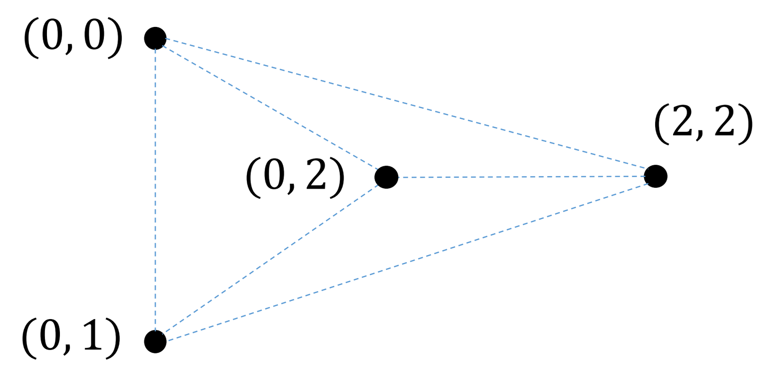
s−distance sets over $\mathbb{F}_q^d$

Let $X \subset \mathbb{F}_q^d$.

- $Q(x) = x^t \cdot x = \sum_{i=1}^d x_i^2$.
- $\Delta_Q(X) = \{Q(x-y) \mid x, y \in X, x \neq y\}$.

A s-distance set over $\mathbb{F}_q^d$ is a set $X \subset \mathbb{F}_q^d$ s.t. $|\Delta_Q(X)| = s$

e.g. $\mathbb{F}_3^2$



2-distance set

> **Theorem 1** (S. 2018+)
> For any s-distance set X and $q \gg 0$,
> $$|X| \leq \frac{2sq^{\frac{d-1}{2}} + 1}{1 - o(1)}.$$

Finite Euclidean graphs

(Medrano et.al. 1995) For $d > 0, a \in \mathbb{F}_q$, *the **finite Euclidian graph*** $E_q(d, a)$ is defined as follows

$$V\left(E_q(d, a)\right) = \mathbb{F}_q^d,\ E\left(E_q(d, a)\right) = \{(x, y) \mid Q(x-y) = a\},$$

where $Q(x) = x^t \cdot x = \sum_{i=1}^d x_i^2$.

> Theorem (Medrano et.al. 1995)
> (1) $E_q(d, a)$ is a regular graph of degree $q^{d-1} + O\left(q^{\frac{d}{2}}\right)$.
> (2) $\lambda\left(E_q(d, a)\right) \leq 2q^{\frac{d-1}{2}}$.

Proof of Theorem 1

> Lemma. (Expander Mixing Lemma)
> If G is a non-bipartite k-regular graph with n vertices, then
> $$\forall X \subset V(G), \left|e_G(X) - \frac{k}{2n}\binom{|X|}{2}\right| \leq \frac{1}{2} \cdot \lambda(G) \cdot |X|$$
> where $e_G(X) = |\{\{x_1, x_2\} \in E(G) \mid x_1, x_2 \in X\}|$

Suppose that X is a s-distance set.

Then,

$$\binom{|X|}{2} = \sum_{a \in \Delta_Q(X)} e_{E_q(d,a)}(X) \leq s\left(\frac{q^{d-1} + O\left(q^{\frac{d}{2}}\right)}{2 \cdot q^d}\binom{|X|}{2} + \frac{1}{2} \cdot 2q^{\frac{d-1}{2}} \cdot |X|\right).$$

From this inequality we get Theorem 1.

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