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On the uniform stationary measure of space-inhomogeneous quantum walks in one dimension

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What is Quantum walk ?

Quantum version of Random walk

▷ Ambainis, A., Bach, E., Nayak, A., Vishwanath, A., Watrous, J.: One-dimensional quantum walks. In: Proceedings of the 33rd Annual ACM Symposium on Theory of Computing, pp.37–49 (2001)

Quantum algorithm

▷ Portugal, R.: Quantum Walks and Search Algorithms, Springer (2013)

Quantum Simulator

▷ Oka, T., Konno, N., Arita, R., Aoki, H.: Breakdown of an electric-field driven system: a mapping to a quantum walk, Phys. Rev. Lett., 94, 100602 (2005)

▷Kitagawa, T.: Topological phenomena in quantum walks: elementary introduction to the physics of topological phases, Quantum Inf. Process., 11, 1107--1148 (2012)

e.t.c. …

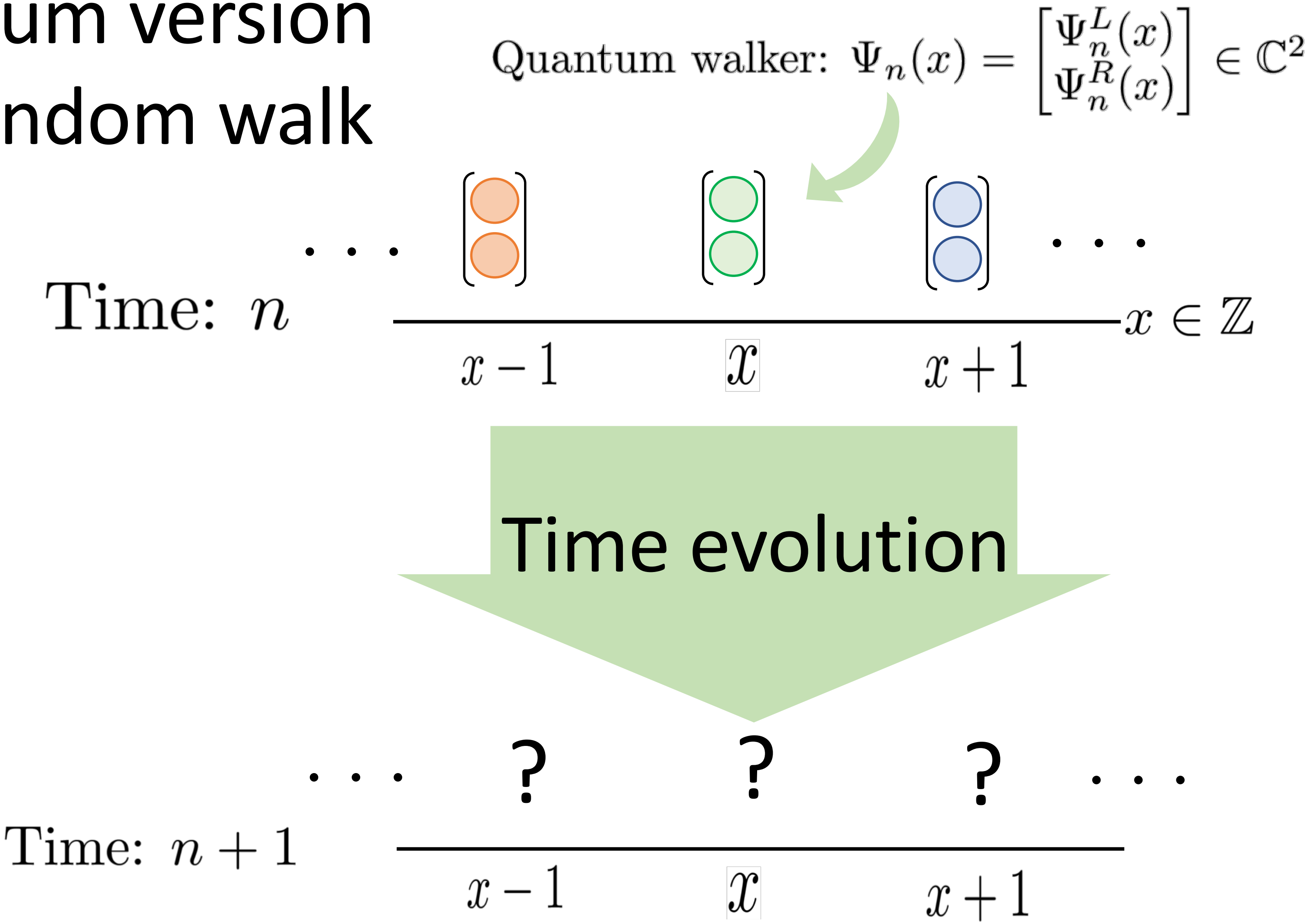
The purpose of this study

From mathematical point of view ,

Quantum walk (QW) VS Random walk (RW)

Quantum walk

Quantum version of Random walk

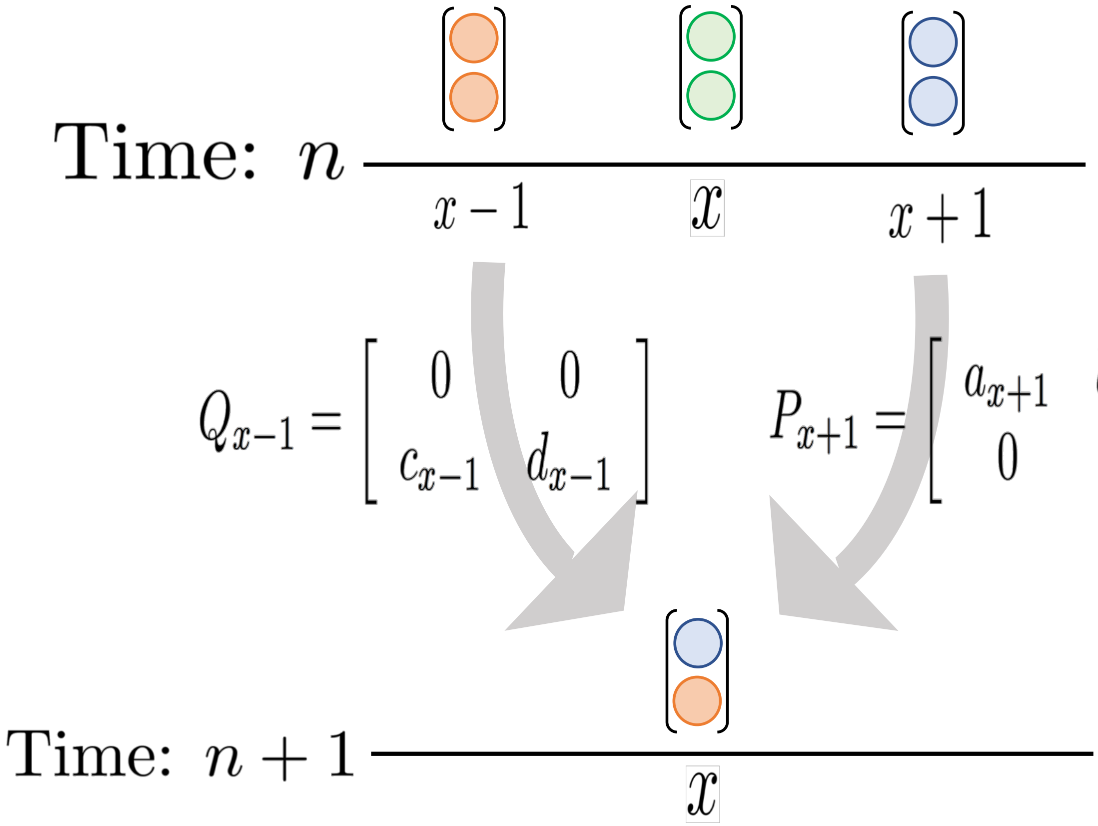


Quantum walk

How ?

U_x: 2×2 Unitary matrix
called Coin matrix or Coin

$$U_x = \begin{bmatrix} a_x & b_x \\ c_x & d_x \end{bmatrix} = \underbrace{\begin{bmatrix} a_x & b_x \\ 0 & 0 \end{bmatrix}}_{P_x} + \underbrace{\begin{bmatrix} 0 & 0 \\ c_x & d_x \end{bmatrix}}_{Q_x}$$



$$Q_{x-1} = \begin{bmatrix} 0 & 0 \\ c_{x-1} & d_{x-1} \end{bmatrix} \qquad P_{x+1} = \begin{bmatrix} a_{x+1} & b_{x+1} \\ 0 & 0 \end{bmatrix}$$

$$\Psi_{n+1}(x) \equiv P_{x+1}\Psi_n(x+1) + Q_{x-1}\Psi_n(x-1)$$

Quantum walk

At each point $\Psi_{n+1}(x) \equiv P_{x+1}\Psi_n(x+1) + Q_{x-1}\Psi_n(x-1)$

$$\Psi_n = {}^T[\, \dots, \Psi_n^L(-1), \Psi_n^R(-1), \Psi_n^L(0), \Psi_n^R(0), \Psi_n^L(+1), \Psi_n^R(+1), \dots \,],$$

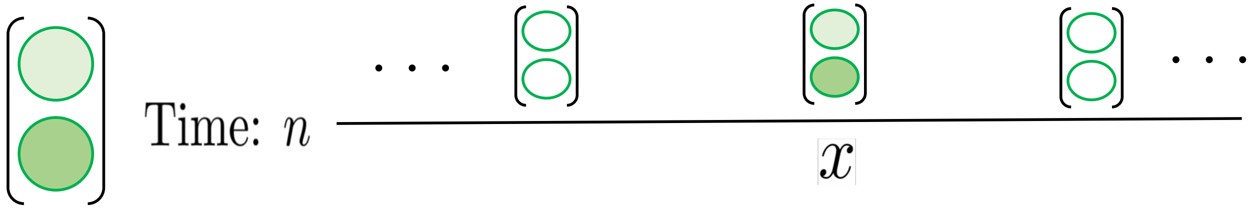
Unitary operator

$$U^{(s)} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & Q_{-2} & O & P_0 & O & \cdots & \cdots \\ \cdots & O & Q_{-1} & O & P_1 & O & \cdots \\ \cdots & \cdots & O & Q_0 & O & P_2 & \cdots \\ \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \text{ with } O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

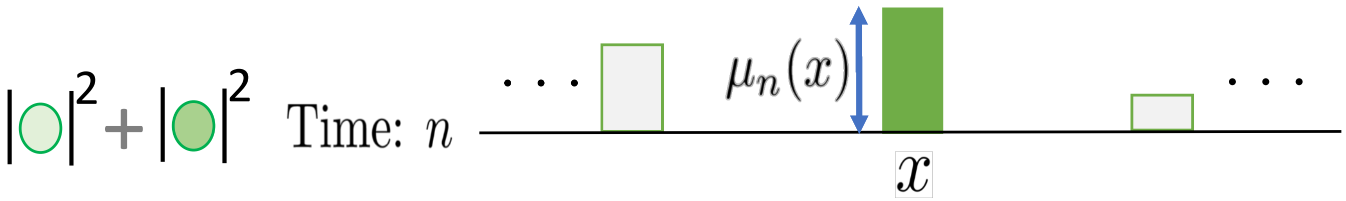
The whole system $\Psi_n = (U^{(s)})^n \Psi_0$ for any $n \geq 0$.

Measures of Quantum walk

Quantum walker: $\Psi_n(x) =$



Norm of Quantum walker
at time n, position x: $\mu_n(x) =$

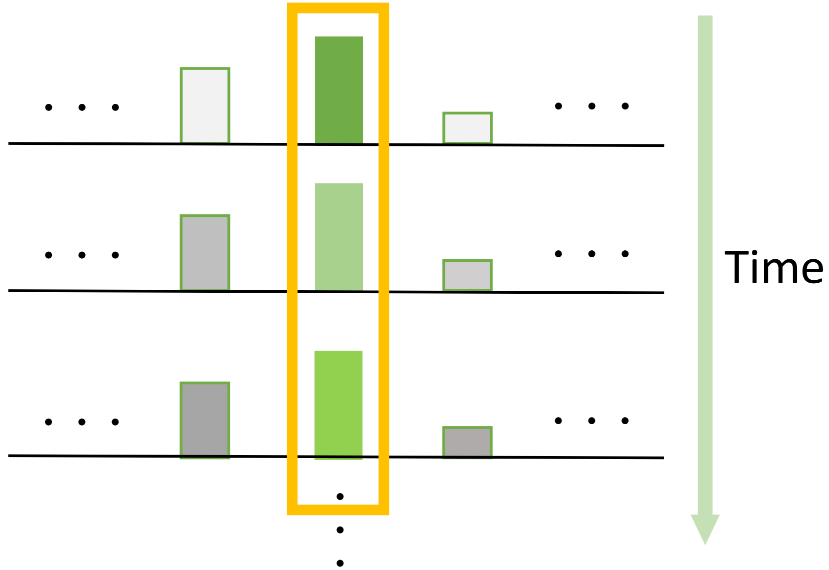


A measure of
Quantum walks

Measures

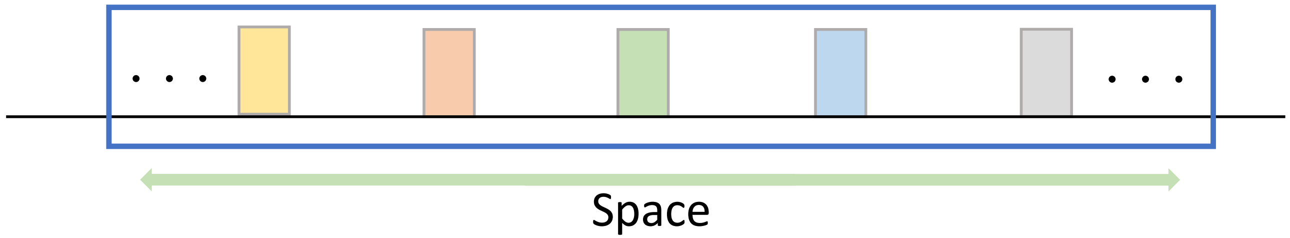
The set of sationary measures: $\mathcal{M}_s(U^{(s)})$

Stationary measure

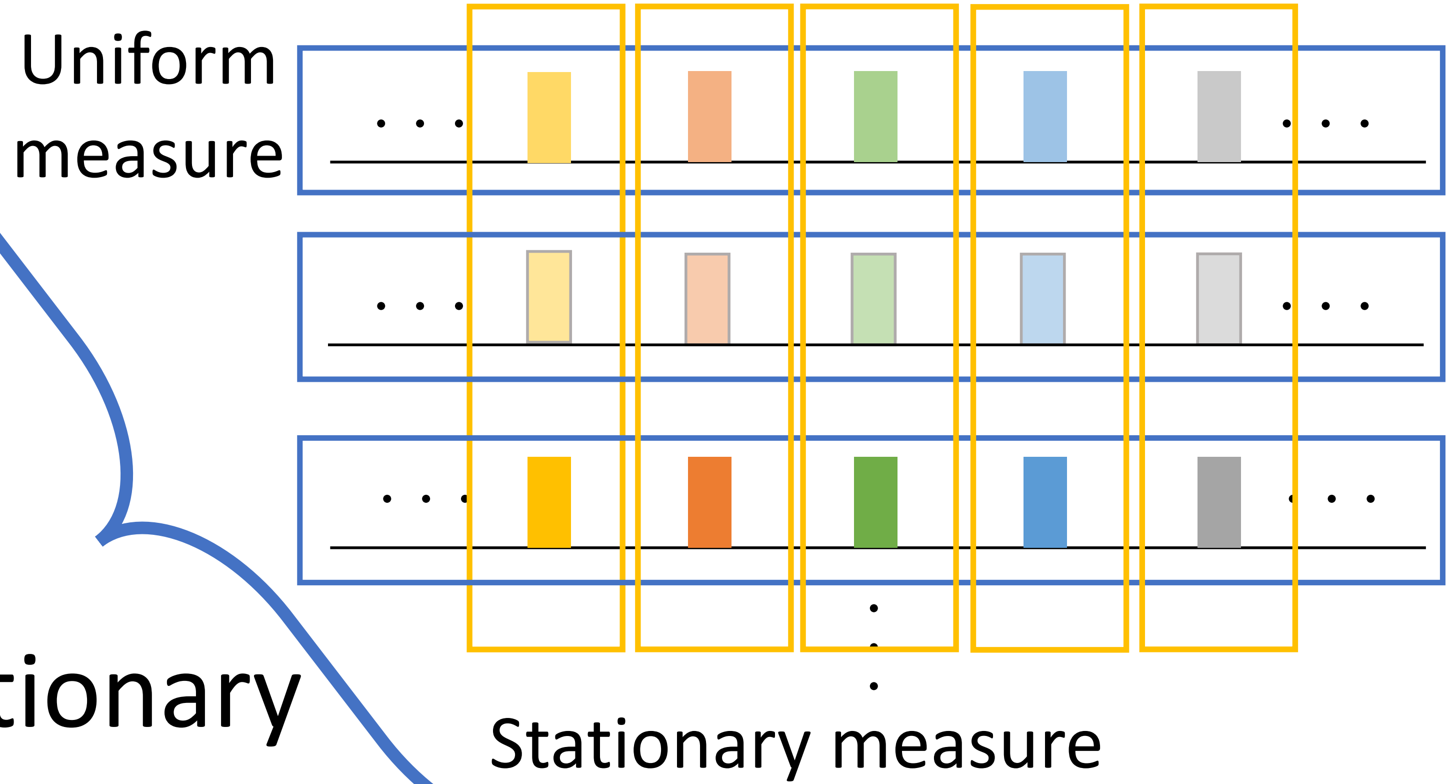


The set of uniform measures: $\mathcal{M}_{unif}$

Uniform measure



Measures



Uniform stationary measure

$$\in \mathcal{M}_{unif} \cap \mathcal{M}_s(U^{(s)})$$

Prior research & this study

Model	Having Uniform stationary measure ?
space-homogeneous QWs	Yes ▷ 1, 2
space-**in**homogeneous QWs	Yes by this study ▷ 3

▷ 1. Konno, N.: The uniform measure for discrete-time quantum walks in one dimension, Quantum Inf. Process., 13, 1103--1125 (2014)
▷ 2. Konno, N., Takei, M.: The non-uniform stationary measure for discrete-time quantum walks in one dimension, Quantum Inf. Comput., bf 15, 1060--1075 (2015)
▷ 3. Ide, Y., Konno, N., Nakayama, D.: On the uniform stationary measure of space-inhomogeneous quantum walks in one dimension, arXiv:1810.12504 (2018)

Main result

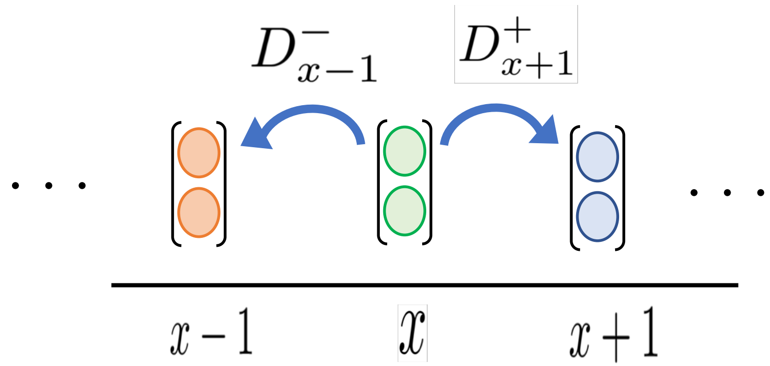
Theorem

QWs defined by the parameters $\{\omega_x\}_{x\in\mathbb{Z}}$

$$U_x = \begin{bmatrix} \cos\theta & e^{i\omega_x}\sin\theta \\ e^{-i\omega_x}\sin\theta & -\cos\theta \end{bmatrix} (\omega_x \in [0,2\pi), \theta \in (0,2\pi))$$

$U_x \in \mathcal{C}_\phi \Leftrightarrow$

Satisfying ${}^\forall x \in \mathbb{Z}$, $\omega_x - \omega_{x-1} = 2\phi \,(\mathrm{mod}\ 2\pi)\ (\phi \in [0,2\pi))$



If $U_x \in \mathcal{C}_\phi$,

$$\Psi(x) = \begin{cases} \prod_{y=1}^{x} D^+_y \Psi(0) & (x \geq 1), \\ \Psi(0) & (x = 0), \\ \prod_{y=-1}^{x} D^-_y \Psi(0) & (x \leq -1), \end{cases}$$

$$D^+_x = \begin{bmatrix} e^{i\phi}\cos\theta & e^{i\alpha_x}\sin\theta \\ e^{-i\alpha_x}\sin\theta & -e^{-i\phi}\cos\theta \end{bmatrix}, \quad D^-_x = \begin{bmatrix} e^{-i\phi}\cos\theta & e^{i\alpha_{x+1}}\sin\theta \\ e^{-i\alpha_{x+1}}\sin\theta & -e^{i\phi}\cos\theta \end{bmatrix}$$

$$\alpha_x = \phi + \omega_{x-1} = \omega_x - \phi \,(\mathrm{mod}\ \ 2\pi)$$

$U_x \in \mathcal{C}_\phi \Rightarrow$ (The measure of Ψ) $\in \mathcal{M}_{unif} \cap \mathcal{M}_s(U^{(s)})$.

Periodicity of coins of QWs

the sequence of coins $\{U_x\}_{x\in\mathbb{Z}}$

$$U_x = \begin{bmatrix} \cos(\theta) & e^{i\omega_x}\sin(\theta) \\ e^{-i\omega_x}\sin(\theta) & -\cos(\theta) \end{bmatrix}$$

$$^{\forall}x \in \mathbb{Z},\ \omega_x - \omega_{x-1} = 2\phi \pmod{2\pi}$$

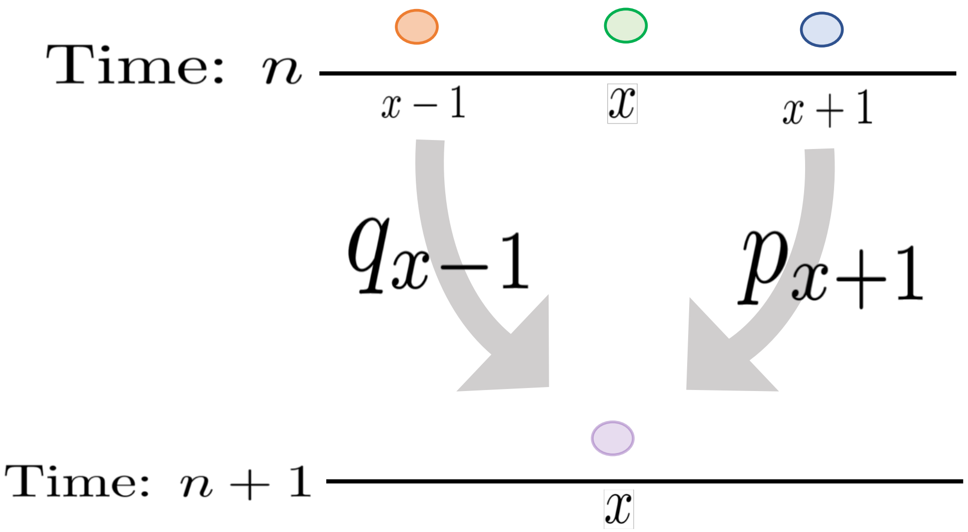
$\phi = \frac{1}{N} \times \pi$	$\{U_x\}_{x\in\mathbb{Z}}$ *has N period*
$\phi = a\pi$	$\{U_x\}_{x\in\mathbb{Z}}$ *has no period*

$N \in \mathbb{N}$

a is an irrational number

Random walk

Random walker: $\mu_n(x) \in \mathbb{R}_{\geq 0}$



$p_x \in [0,1] \quad q_x = 1 - p_x$

At each point

$$\mu_n(x) = p_{x+1}\mu_{n-1}(x+1) + q_{x-1}\mu_{n-1}(x-1)$$

$$\mu_n = {}^T[\ \dots\ , \mu_n(-1), \mu_n(0), \mu_n(+1)\ \dots\ ],$$

$$P^{(s)} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & q_{-2} & 0 & p_0 & 0 & \cdots & \cdots \\ \cdots & 0 & q_{-1} & 0 & p_1 & 0 & \cdots \\ \cdots & \cdots & 0 & q_0 & 0 & p_2 & \cdots \\ \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

The whole system $\quad \mu_n = P^{(s)}\mu_0$

➡ $M_s(P^{(s)})$: the set of stationary measures of the RW

The sequence of coins $\quad \{p_x\}_{x\in\mathbb{Z}}$

Difference between QW and RW

Having uniform stationary measure

The number of periodicity of coins	1	2	3	4	・・・	∞
RW	○	○	×	×	・・・	×
QW	○	○	○	○	・・・	○

Main references

Ambainis, A., Bach, E., Nayak, A., Vishwanath, A., Watrous, J.: One-dimensional quantum walks. In: Proceedings of the 33rd Annual ACM Symposium on Theory of Computing, pp.37--49 (2001)

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