Random Planar Map $O(n)$ Model Nesting & CLE in Liouville Quantum Gravity

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Random Planar Map

A random triangulation [Courtesy of N. Curien].
Random Planar Map

A random triangulation [Courtesy of N. Curien].

Continuum limit: The Brownian Map [Le Gall ’11; Miermont ’11]
Random Planar Map & Conformal Map

[Courtesy of N. Curien]

Left: A random triangulation of the sphere. Right: Conformal map to the sphere.

In the continuum scaling limit: Liouville Quantum Gravity

A.M. Polyakov ’81
Random Planar Map & Statistical Model

Percolation hulls [Courtesy of N. Curien].
Liouville QG
Random Measure

$\mu = \left\langle e^{\gamma h} \right\rangle$
Gaussian Free Field (GFF)

Distribution $h$ with Gaussian weight $\exp\left[-\frac{1}{2}(h, h)_{\nabla}\right]$, and Dirichlet inner product in domain $D$

$$(f_1, f_2)_{\nabla} := (2\pi)^{-1} \int_D \nabla f_1(z) \cdot \nabla f_2(z) dz$$

$$= \text{Cov}((h, f_1)_{\nabla}, (h, f_2)_{\nabla})$$
**Liouville Quantum Measure**

\[
\mu_\gamma := \lim_{\varepsilon \to 0} \exp[\gamma h_\varepsilon(z)] \varepsilon^{\gamma^2/2} dz,
\]

where \( h_\varepsilon(z) \) is the GFF average on a circle of radius \( \varepsilon \); converges weakly for \( \gamma < 2 \) to a random measure, denoted by \( \mu_\gamma = e^{\gamma h(z)} dz \), and singular w. r. t. Lebesgue measure.

[Høegh-Krohn ’71; Kahane ’85; D. & Sheffield ’11]

For \( \gamma = 2 \), the **renormalized** one,

\[
\sqrt{\log(1/\varepsilon)} \left[ \exp[\gamma h_\varepsilon(z)] \varepsilon^{\gamma^2/2} \right]_{\gamma=2} \ dz,
\]

converges, as \( \varepsilon \to 0 \), to a positive **non-atomic** random measure.

[D., Rhodes, Sheffield, Vargas ’14]
Scaling Exponents of (Random) Fractals

SAW in half plane - 1,000,000 steps

Probabilities & Hausdorff Dimensions (e.g., SLE_κ)

\[ P \asymp \varepsilon^{2x}, \quad \tilde{P} \asymp \varepsilon^{\tilde{x}} , \quad d = 2 - 2x \quad (= 1 + \kappa/8) \]

δ-Quantum Ball: \[ P \asymp \delta^{\Delta}, \quad \tilde{P} \asymp \tilde{\delta}^{\tilde{\Delta}} \]
Knizhnik, Polyakov, Zamolodchikov ’88

$x$ and $\Delta$ ($\tilde{x}$ and $\tilde{\Delta}$) are related by the KPZ formula

$$x = U_\gamma(\Delta) := \left(1 - \frac{\gamma^2}{4}\right) \Delta + \frac{\gamma^2}{4} \Delta^2$$

Kazakov ’86; D. & Kostov ’88 [Random matrices]

David; Distler & Kawai ’88 [Liouville field theory]

KPZ Theorem – D. & Sheffield ’11

Benjamini & Schramm ’09; Rhodes & Vargas ’11 [Hausdorff dimension]

David & Bauer ’09; Berestycki, Garban, Rhodes, Vargas ’14 [Heat kernel]
$O(n)$-Loop Model on a Random Planar Map

Disk triangulation and local weights ($\alpha = 1$).

$$Z_\ell = \sum_C u^{V(C)} w(C), \quad w(C) = n^L g^{T_1} h^{T_2};$$

- Sum over all configurations $C$ of a disk of fixed perimeter $\ell$
- $u$ auxiliary weight per vertex, $V(C)$ total number of vertices (volume)
- $T_1, T_2$ numbers of empty or occupied triangles
- number of loops $L$ of $C$ weighted by $n \in [0, 2]$. 
Phase diagram of the $O(n)$-loop model ($n \in [0, 2]$) on a random map. For $u = 1$, a line of critical points separates the subcritical and supercritical phases. Critical points may be in three different universality classes: generic, dilute and dense.
**Random Map Nesting Theorem** [Borot, Bouttier, D. ’16]

Fix \((g, h, \alpha)\) and \(n \in (0, 2)\) such that the model with bending energy reaches a **dilute** or **dense** critical point for the vertex weight \(u = 1\). In the ensemble of random pointed disks of volume \(V\) and perimeter \(L\), the probability distribution of the number \(\mathcal{N}\) of separating loops between the marked point and the boundary behaves as:

\[
P\left[\mathcal{N} = \frac{c \ln V}{\pi} p \bigg| V, L = \ell\right] \sim (\ln V)^{-\frac{1}{2}} V^{-\frac{c}{2}} J(p) \quad \text{(sphere)},
\]

\[
P\left[\mathcal{N} = \frac{c \ln V}{2\pi} p \bigg| V, L = V^{\frac{c}{2}} \ell\right] \sim (\ln V)^{-\frac{1}{2}} V^{-\frac{c}{2\pi}} J(p) \quad \text{(disk)},
\]

where \(\ell > 0\) is fixed, and \(\ln V \gg p\), and:

\[
J(p) = p \ln \left(\frac{2}{n} \frac{p}{\sqrt{1 + p^2}}\right) + \arccot(p) - \arccos\left(\frac{n}{2}\right).
\]

with \(c = 1\) (**dilute**), \(c = 1/[1 - \frac{1}{\pi} \arccos\left(\frac{n}{2}\right)]\) (**dense**), \(c \in [1, 2]\), \(n \in [0, 2]\).

See also Borot, Garcia-Failde ’16; Chen, Curien, Maillard ’17; Budd ’18
Large Deviations Function

\[ J(p) \]

\( J(p) \) for \( n = 1 \) (Ising & Percolation), \( n = \sqrt{2} \) (FK Ising), \( n = \sqrt{3} \) (3-state Potts), \( n = 2 \) (4-state Potts & CLE\(_4\)).
Conformal Loop Ensemble (CLE)

[Sheffield ’09, Sheffield & Werner ’12]

The critical $O(n)$-model on a regular planar lattice is predicted to converge in the continuum scaling limit to $\text{SLE}_\kappa/\text{CLE}_\kappa$, for

$$n = -2 \cos \left( \frac{4\pi}{\kappa} \right), \quad n \in (0, 2], \quad \begin{cases} \kappa \in (8/3, 4], \text{ dilute phase} \\ \kappa \in [4, 8), \text{ dense phase}, \end{cases}$$

(Loop-erased random walk & spanning trees [Lawler, Schramm, Werner], Ising & percolation [Smirnov], GFF contour lines [Schramm-Sheffield].)

On a random planar map: random measure conjectured to be, after uniformization, (a form of) the Liouville quantum measure $\mu_{\gamma}$ for

$$\gamma = \min \{ \sqrt{\kappa}, 4/\sqrt{\kappa} \},$$

and independent GFF & CLE (KPZ ’88, Q. Z., Q.I. [Sheffield ’10], M.o.T. [D., Miller, Sheffield ’14], $\gamma = \sqrt{8/3}$ [Miller–Sheffield ’15, ’17])
\( \mathcal{N}_z(\varepsilon) \) is the number of nested loops of a \( \mathrm{CLE}_\kappa \), \( \kappa \in (8/3, 8) \) surrounding the ball \( B(z, \varepsilon) \) in the unit disk.
Extreme nesting in CLE [Miller, Watson & Wilson ’14]
Let $\mathcal{N}_z(\varepsilon)$ be the number of loops of a CLE$_\kappa$, $\kappa \in (8/3, 8)$ surrounding the ball $B(z, \varepsilon)$, and $\Phi_\nu$ the set of points $z$ where

$$\lim_{\varepsilon \to 0} \frac{\mathcal{N}_z(\varepsilon)}{\ln(1/\varepsilon)} = \nu.$$ 

$$\dim_{\mathcal{H}} \Phi_\nu = 2 - \gamma_\kappa(\nu)$$

$$\gamma_\kappa(\nu) = \nu \Lambda^*_\kappa(1/\nu), \nu \geq 0; \quad \Lambda^*_\kappa(x) := \sup_{\lambda \in \mathbb{R}} (\lambda x - \Lambda_\kappa(\lambda))$$

$$\Lambda_\kappa(\lambda) = \ln \left( \frac{-\cos(4\pi/\kappa)}{\cos \left( \pi \sqrt{(1 - 4/\kappa)^2 + 8\lambda/\kappa} \right)} \right)$$

Moment generating function of the loop log-conformal radius [Cardy & Ziff ’02; Kenyon & Wilson ’04; Schramm, Sheffield & Wilson ’09]
Conformal Loop Ensemble \( \text{CLE}_\kappa, \kappa \in (8/3, 8) \)

\( \mathcal{U} \) the connected component containing 0 in the complement \( \mathbb{D} \setminus \mathcal{L} \) of the largest loop \( \mathcal{L} \) surrounding 0 in \( \mathbb{D} \). Cumulant generating function of \( T = -\ln(\text{CR}(0, \mathcal{U})) \) [Schramm, Sheffield, Wilson ’09]

\[
\Lambda_\kappa(\lambda) := \ln \mathbb{E} \left[ e^{\lambda T} \right] = \ln \left( \frac{-\cos(4\pi/\kappa)}{\cos \left( \pi \left[ \left(1 - \frac{4}{\kappa}\right)^2 + \frac{8\lambda}{\kappa} \right]^{1/2} \right)} \right), \lambda \in (-\infty, 1 - \frac{2}{\kappa} - \frac{3\kappa}{32}).
\]
Large Deviations Function

\[ \frac{(2\pi)^2}{\kappa} \gamma_{\kappa}(\nu) \]

CLE\(_\kappa\) nesting large deviations function, \(\gamma_{\kappa}(\nu)/\kappa\),
for \(\kappa = 3\) or 6 (Ising / Percolation, \(n = 1\)), \(\kappa = 16/3\) (FK-Ising, \(n = \sqrt{2}\)),
\(\kappa = 25/4\) (3-state Potts, \(n = \sqrt{3}\)), \(\kappa = 4\) (GFF contour lines, \(n = 2\))
Multifractal Spectrum

\[ 2 - \gamma_\kappa(\nu) \]

\text{CLE}_\kappa \text{ nesting Hausdorff dimension, } \dim_{\mathcal{H}} \Phi_\nu = 2 - \gamma_\kappa(\nu),

for \( \kappa = 3 \) (Ising), \( \kappa = 4 \) (GFF contour lines), \( \kappa = 6 \) (Percolation).
Large Deviations

Euclidean case: for a ball of radius $\varepsilon$

$$\mathbb{P}(\mathcal{N}_z \approx v \ln(1/\varepsilon) \mid \varepsilon) = \mathbb{P}(\mathcal{N}_z \approx vt \mid t) \approx \varepsilon^{\gamma(t)} = \exp[-t\gamma(t)].$$

Liouville Quantum Gravity:

$$t := -\ln \varepsilon; \quad A := -\gamma^{-1} \ln \delta, \quad \delta := \int_{B(z,\varepsilon)} \mu \gamma \quad \text{(quantum ball)}$$

**Conditioned on $\delta$, hence $A$, perform the convolution**

$$\mathbb{P}_Q(\mathcal{N}_z \mid A) := \int_0^\infty \mathbb{P}(\mathcal{N}_z \mid t) P(t \mid A) dt,$$

where $P(t \mid A)$ is the probability distribution of the random Euclidean log-radius $t$, given the quantum log-radius $A$. 
Probability Distribution [D.– Sheffield ’09]

\[ AP_A(t) \]

\[ \gamma = \sqrt{8/3} \ [A = 2; 20; 200] \]

\[ P(t | A) = \frac{A}{\sqrt{2\pi t^3}} \exp \left[ -\frac{1}{2t} (A - a_\gamma t)^2 \right] \]

\[ t = -\ln \varepsilon, \ A = -\gamma^{-1} \ln \delta, \ \delta = \int_{B(z,\varepsilon)} \mu_\gamma \]

\[ a_\gamma := 2/\gamma - \gamma/2 \]
Quantum Large Deviations

\[ t = -\ln \varepsilon, \quad A = -\gamma^{-1} \log \delta \text{ (quantum ball),} \]

\[ \mathcal{N} \approx -\nu \ln \varepsilon = \nu t, \quad \mathcal{N} \approx -p \ln \delta = \gamma p A, \]

which implies \( \nu t = \gamma p A \). The above convolution then yields, for \( A \to +\infty \),

\[
P_Q (\mathcal{N} \approx \gamma p A \mid A) \approx \int_0^\infty \frac{dt A}{\sqrt{2\pi t^3}} \exp \left( -\frac{(A - a\gamma t)^2}{2t} - \gamma \kappa (\nu) t \right)
\]

\[
\approx \exp \left[ -A \Theta(p) \right] \quad \text{(saddle point at constant } \nu t) \]

\( \Theta(p) \) is the large deviations function for the loop number around a \( \delta \)-quantum ball to scale as \( p \log(1/\delta) \).
Legendre Transform & KPZ

In the plane, the Legendre transform gave

\[ \gamma_\kappa(v) = \lambda - v \Lambda_\kappa(\lambda), \quad \frac{1}{v} = \frac{\partial \Lambda_\kappa(\lambda)}{\partial \lambda}. \]

In Liouville Quantum Gravity

\[ \Theta(p) = U^{-1}_\gamma(\lambda/2) - p \Lambda_\kappa(\lambda), \quad \frac{1}{p} = \frac{\partial \Lambda_\kappa(\lambda)}{\partial U^{-1}_\gamma(\lambda/2)}, \]

where \( U^{-1}_\gamma(\lambda/2) := (\sqrt{a_\gamma^2 + 2\lambda} - a_\gamma)/\gamma \) is the inverse KPZ function, with

\[ \gamma = \min \left\{ \sqrt{\kappa}, 4/\sqrt{\kappa} \right\}, \quad a_\gamma = 2/\gamma - \gamma/2. \]
**Theorem [Borot, Bouttier, D. ’16]**

In Liouville quantum gravity, the cumulant generating function $\Lambda_\kappa$, with $\kappa \in (8/3, 8)$, is transformed into the quantum one, $\Lambda_Q^\kappa := \Lambda_\kappa \circ 2U_\gamma$, where $U_\gamma(\lambda) := \left(1 - \frac{\gamma^2}{4}\right)\lambda + \frac{\gamma^2}{4}\lambda^2$ is the KPZ function for $\gamma = \min\{\sqrt{\kappa}, 4/\sqrt{\kappa}\}$.

Its Legendre-Fenchel transform is

$$\Lambda_Q^\kappa(\lambda := \sup_{\lambda \in \mathbb{R}} \left(\lambda x - \Lambda_Q^\kappa(\lambda)\right).$$

The quantum nesting distribution in the disk is then, for $\delta \to 0$,

$$\mathbb{P}_Q(\mathcal{N}_z \approx p \ln(1/\delta) \mid \delta) \sim \delta^{\Theta(p)},$$

where

$$\Theta(p) = \begin{cases} 
  p \Lambda_Q^\kappa(1/p), & \text{if } p > 0 \\
  3/4 - 2/\kappa & \text{if } p = 0 \text{ and } \kappa \in (8/3, 4] \\
  1/2 - \kappa/16 & \text{if } p = 0 \text{ and } \kappa \in [4, 8). 
\end{cases}$$
**Corollary** [Borot, Bouttier, D. ’16]  

The **quantum** generating function associated with CLE$_\kappa$ nesting is, for $\kappa \in \left( \frac{8}{3}, 8 \right)$

$$
\Lambda^Q_{\kappa}(\lambda) = \Lambda_{\kappa} \circ 2U_{\gamma}(\lambda) = \ln \left( \frac{\cos \left[ \pi \left( 1 - \frac{4}{\kappa} \right) \right]}{\cos \left[ \pi \left( \frac{2\lambda}{c} + \left| 1 - \frac{4}{\kappa} \right| \right) \right]} \right), \quad c = \max\{1, \kappa/4\},
$$

$\lambda \in \left[ \frac{1}{2} - \frac{2}{3\kappa}, \frac{3}{4} - \frac{1}{8\kappa} \right]$ for $\kappa \in \left( \frac{8}{3}, 4 \right)$; \quad $\lambda \in \left[ \frac{1}{2} - \frac{\kappa}{8}, \frac{1}{2} - \frac{\kappa}{16} \right]$ for $\kappa \in [4, 8)$.

- **The KPZ relation**, which usually concerns scaling dimensions, acts here on a conjugate variable in a Legendre transform.
- **The composition map** $\Lambda_{\kappa} \rightarrow \Lambda^Q_{\kappa} = \Lambda_{\kappa} \circ 2U_{\gamma}$ to go from Euclidean geometry to Liouville quantum geometry is fairly general.
Theorem [Borot, Bouttier, D. ’16]

The quantum nesting probability of \( \text{CLE}_\kappa \) in a simply connected domain, for the number \( \mathcal{N}_z \) of loops surrounding a ball centered at \( z \) and conditioned to have a given Liouville quantum measure \( \delta \), has the large deviations form,

\[
\mathbb{P}_Q \left( \mathcal{N}_z \approx \frac{cp}{2\pi} \ln(1/\delta) \bigg| \delta \right) \approx \delta \frac{c}{2\pi} J(p), \quad \delta \to 0,
\]

\[
\Theta \left( \frac{cp}{2\pi} \right) = \frac{c}{2\pi} J(p),
\]

where \( c \) and \( J \) are the same as in the combinatorial result for the critical \( O(n) \) model in the scaling limit of large random maps.
Quantum Large Deviations Function

\[ \Theta(p) = \frac{c}{2\pi} J \left( \frac{c}{2\pi} p \right) \]

\[ \Theta(p) \] for \( \kappa = 3 \) (Ising), \( \kappa = 4 \) (GFF contour lines), \( \kappa = 6 \) (Percolation).
Quantum Multifractal Spectrum

$$1 - \Theta(p)$$

$1 - \Theta(p)$ for $\kappa = 3$ (Ising), $\kappa = 4$ (GFF contour lines), $\kappa = 6$ (Percolation).

**Quantum Hausdorff Dimension** for $p$-nesting points: $D_{\mathcal{H}} (1 - \Theta(p))$,
with $D_{\mathcal{H}}$ Hausdorff dimension of the $\gamma$-Liouville quantum surface.
**CLE on the Riemann sphere** [Kemppainen & Werner ’14]

**Theorem** [Borot, Bouttier, D. ’16]

The nesting probability in $\text{CLE}_\kappa(\hat{C})$ between two balls of radius $\varepsilon_1$ and $\varepsilon_2$ and centered at two distinct punctures, has the large deviations form,

$$
\mathbb{P}^{\hat{C}}\left[ N(\varepsilon_1, \varepsilon_2) \approx \nu \ln(1/(\varepsilon_1\varepsilon_2)) \right] \asymp (\varepsilon_1\varepsilon_2)^{\gamma_\kappa(\nu)}, \quad \nu \geq 0, \quad \varepsilon_1, \varepsilon_2 \to 0,
$$

where $\gamma_\kappa(\nu)$ is the large deviations function of the disk topology.

**Corollary**

For two balls of same radius $\varepsilon$,

$$
\mathbb{P}^{\hat{C}}\left[ N(\varepsilon, \varepsilon) \approx \nu \ln(1/\varepsilon) \right] \asymp \varepsilon^{\tilde{\gamma}_\kappa(\nu)}, \quad \nu \geq 0, \quad \varepsilon \to 0,
$$

where $\tilde{\gamma}_\kappa(\nu)$ is related to the disk large deviations function by

$$
\tilde{\gamma}_\kappa(\nu) = 2\gamma_\kappa(\nu/2).
$$
Quantum Riemann sphere

**Theorem [Borot, Bouttier, D. ’16]**

On the quantum sphere $\hat{\mathbb{C}}$, the large deviations function $\hat{\Theta}$ which governs the nesting probability between two non-overlapping $\delta$-quantum balls,

$$\mathbb{P}_{\hat{\mathbb{C}}} (\mathcal{N} \approx p \ln(1/\delta) \mid \delta) \approx \delta^{\hat{\Theta}(p)}, \quad \delta \to 0,$$

is related to the $\Theta$ function for the disk topology by

$$\hat{\Theta}(p) = 2\Theta(p/2),$$

so that

$$\mathbb{P}_{\hat{\mathbb{C}}} \left( \mathcal{N} \approx \frac{cp}{\pi} \ln(1/\delta) \mid \delta \to 0 \right) \approx \delta^{\frac{c}{\pi}J(p)},$$

where $c$ and $J$ are the same as before.

**Perfect matching** of LQG results for CLE$_\kappa$ with those for the $O(n)$ model on a random planar map, with the correspondence $\delta \leftrightarrow 1/V$, with $\delta \to 0, V \to +\infty$. 