Random Planar Map $O(n)$ Model Nesting & CLE in Liouville Quantum Gravity

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Random Planar Map

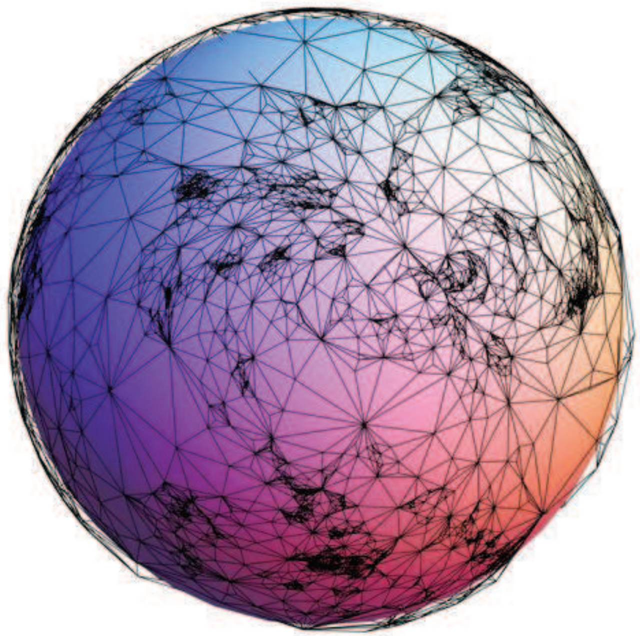
A random triangulation [Courtesy of N. Curien].

Random Planar Map

A random triangulation [Courtesy of N. Curien].

Continuum limit: **The Brownian Map** *[Le Gall '11; Miermont '11]*

Random Planar Map & Conformal Map



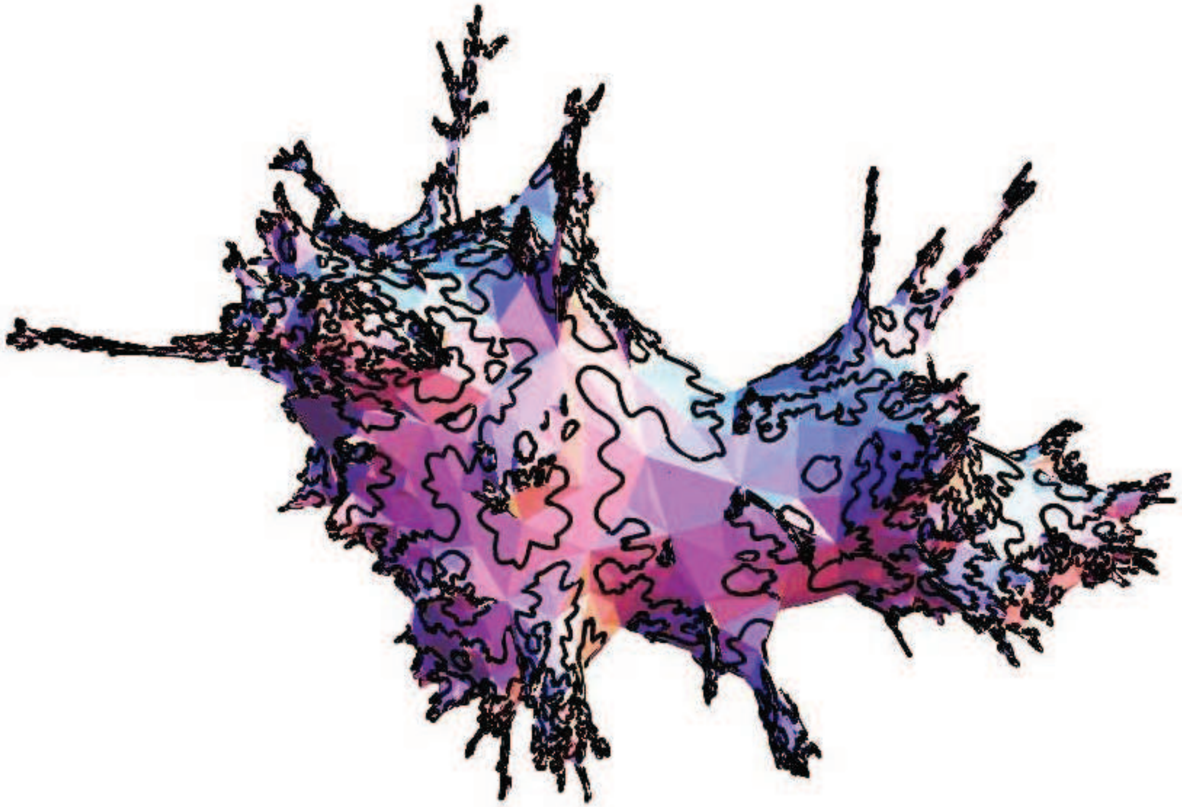
[Courtesy of N. Curien]

Left: A random triangulation of the sphere. Right: Conformal map to the sphere.

In the continuum scaling limit: **Liouville Quantum Gravity**

A.M. Polyakov '81

Random Planar Map & Statistical Model



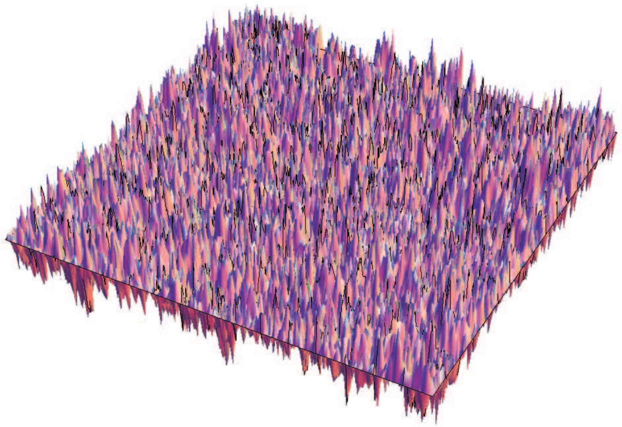
Percolation hulls [Courtesy of N. Curien].

Liouville QG Random Measure

$$\mu = \text{“}e^{\gamma h}dz\text{”}$$

◊

Gaussian Free Field (GFF)



[Courtesy of J. Miller]

Distribution h with *Gaussian weight* $\exp\left[-\frac{1}{2}(h,h)_\nabla\right]$, and **Dirichlet inner product** in domain D

$$\begin{aligned}(f_1, f_2)_\nabla &:= (2\pi)^{-1}\int_D \nabla f_1(z)\cdot\nabla f_2(z)dz \\ &= \mathrm{Cov}\big((h,f_1)_\nabla,(h,f_2)_\nabla\big)\end{aligned}$$

LIOUVILLE QUANTUM MEASURE

$$\mu_\gamma := \lim_{\varepsilon\to 0} \exp\left[\gamma h_\varepsilon(z)\right] \varepsilon^{\gamma^2/2} dz,$$

where $h_\varepsilon(z)$ *is the GFF average on a circle of radius* ε*;*
converges weakly for $\gamma < 2$ *to a random measure, denoted by*
$\mu_\gamma = e^{\gamma h(z)} dz$*, and singular w. r. t. Lebesgue measure.*

[Høegh-Krohn '71; Kahane '85; D. & Sheffield '11]

For $\gamma = 2$*, the* **renormalized** *one,*

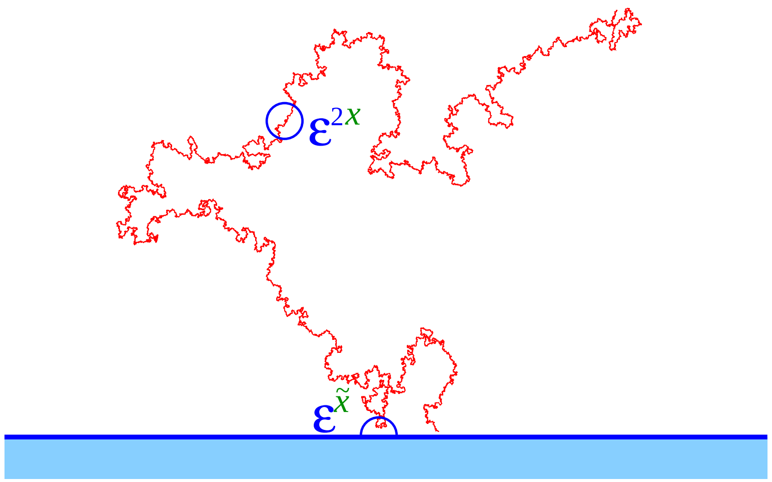
$$\sqrt{\log(1/\varepsilon)} \left[\exp\left[\gamma h_\varepsilon(z)\right] \varepsilon^{\gamma^2/2}\right]_{\gamma=2} dz,$$

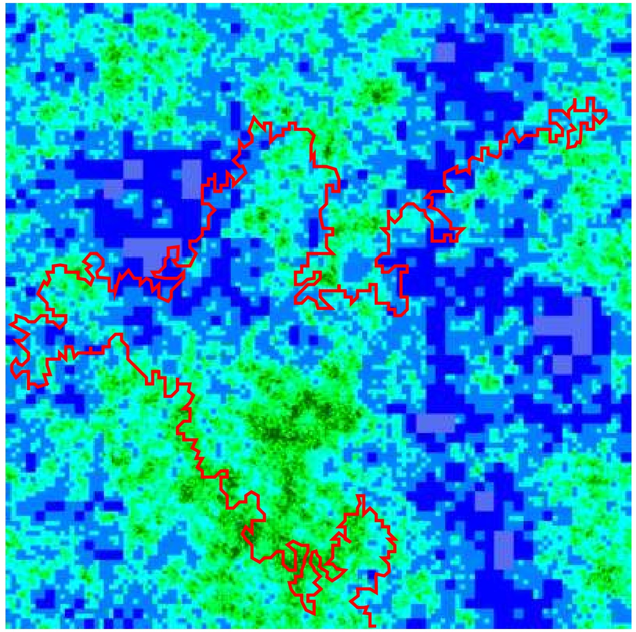
converges, as $\varepsilon \to 0$*, to a positive non-atomic random measure.*

[D., Rhodes, Sheffield, Vargas '14]

Scaling Exponents of (Random) Fractals

SAW in half plane - 1,000,000 steps





[Courtesy of T. Kennedy & J. Miller]

Probabilities & Hausdorff Dimensions (e.g., SLE_κ)

$\mathbb{P} \asymp \varepsilon^{2x}, \quad \widetilde{\mathbb{P}} \asymp \varepsilon^{\tilde{x}}, \qquad d = 2 - 2x \qquad (= 1 + \kappa/8)$

δ-*Quantum Ball:* $\quad \mathbf{P} \asymp \delta^{\Delta}, \quad \widetilde{\mathbf{P}} \asymp \delta^{\widetilde{\Delta}}$

Knizhnik, Polyakov, Zamolodchikov '88

x and Δ ($\tilde{x}$ and $\tilde{\Delta}$) are related by the **KPZ formula**

$$x = U_\gamma(\Delta) := \left(1 - \frac{\gamma^2}{4}\right)\Delta + \frac{\gamma^2}{4}\Delta^2$$

Kazakov '86; D. & Kostov '88 [Random matrices]

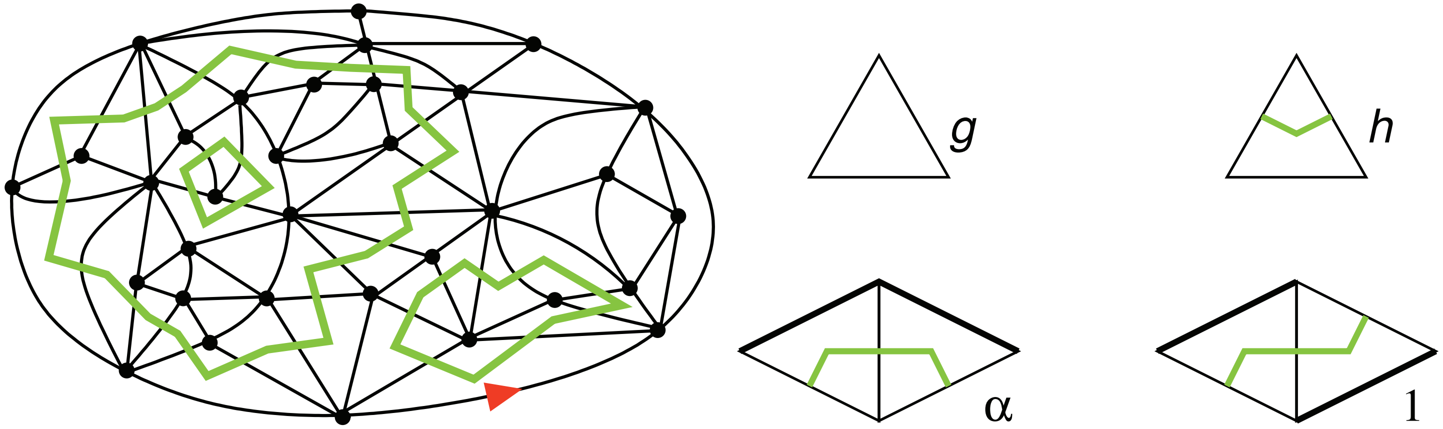
David; Distler & Kawai '88 [Liouville field theory]

KPZ Theorem – *D. & Sheffield '11*

Benjamini & Schramm '09; Rhodes & Vargas '11 [Hausdorff dimension]

David & Bauer '09; Berestycki, Garban, Rhodes, Vargas '14 [Heat kernel]

$O(n)$-Loop Model on a Random Planar Map

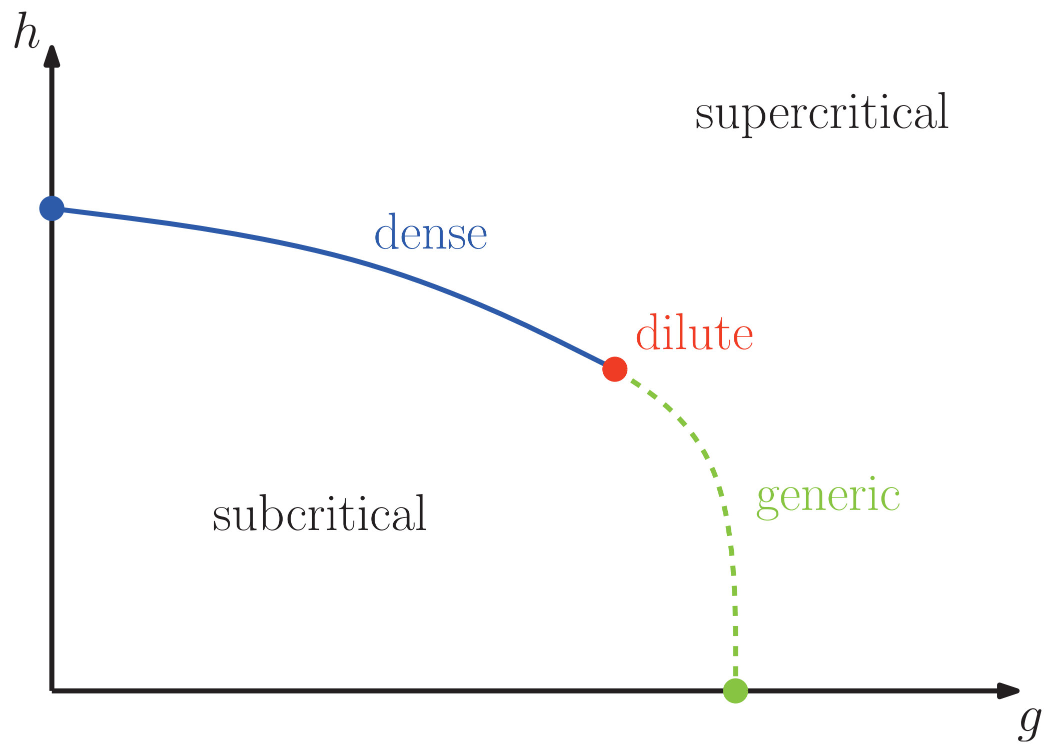


Disk triangulation and local weights ($\alpha = 1$).

$$Z_\ell = \sum_{\mathcal{C}} u^{V(\mathcal{C})} w(\mathcal{C}), \; w(C) = n^{\mathcal{L}} g^{T_1} h^{T_2};$$

- Sum over all **configurations** $\mathcal{C}$ of a **disk** of fixed perimeter ℓ
- u auxiliary weight per **vertex**, $V(\mathcal{C})$ total number of vertices (*volume*)
- T_1, T_2 numbers of empty or **occupied** triangles
- number of **loops** $\mathcal{L}$ of $\mathcal{C}$ weighted by $n \in [0,2]$.

Phase Diagram



Phase diagram of the $O(n)$-loop model ($n \in [0,2]$) on a random map.
For $u = 1$, a line of critical points separates the subcritical and supercritical phases. Critical points may be in three different **universality classes***:* **generic***,* **dilute** *and* **dense***.*

Random Map Nesting Theorem *[Borot, Bouttier, D. '16]*
Fix (g, h, α) *and* $n \in (0, 2)$ *such that the model with bending energy reaches a* **dilute** *or* **dense** *critical point* for the vertex weight $u = 1$. In the ensemble of random pointed disks of volume V and perimeter L, the probability distribution of the number $\mathcal{N}$ of separating loops between the marked point and the boundary behaves as:

$$\mathbb{P}\left[\mathcal{N} = \frac{c \ln V}{\pi} p \,\middle|\, V, L = \ell\right] \dot{\sim} (\ln V)^{-\frac{1}{2}} V^{-\frac{c}{\pi} J(p)} \quad \text{(sphere)},$$

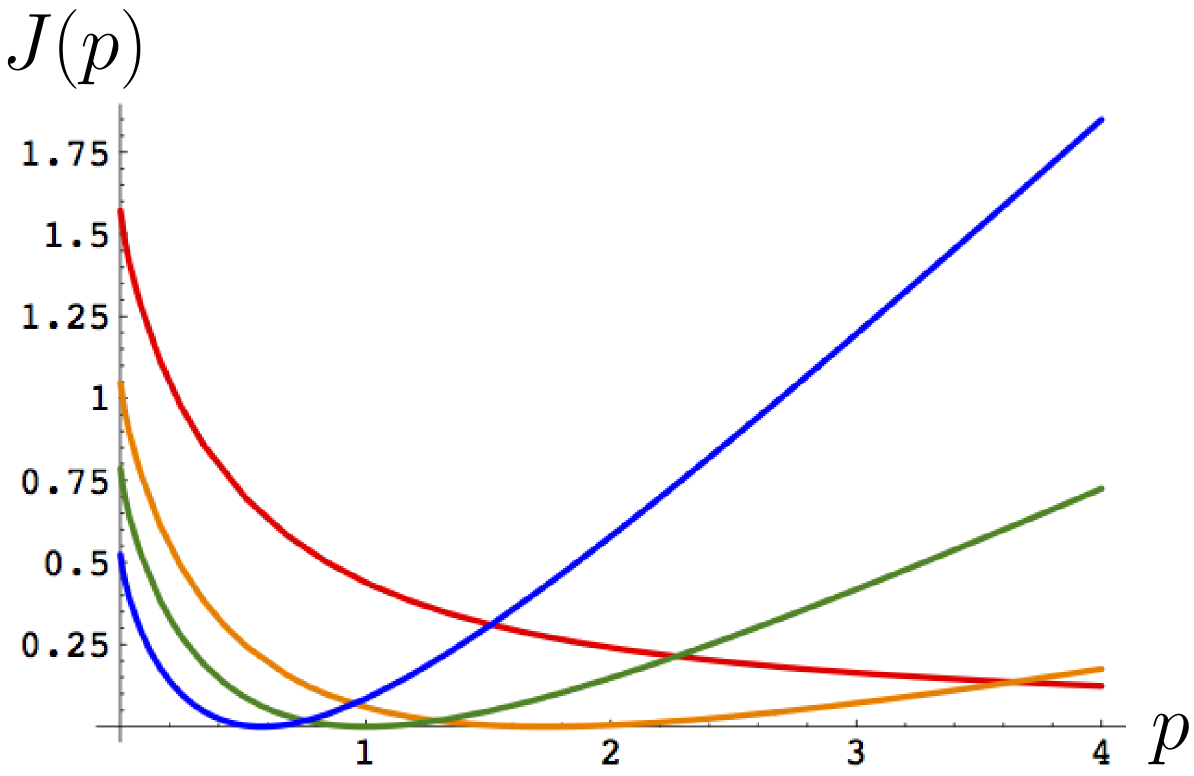
$$\mathbb{P}\left[\mathcal{N} = \frac{c \ln V}{2\pi} p \,\middle|\, V, L = V^{\frac{c}{2}} \ell\right] \dot{\sim} (\ln V)^{-\frac{1}{2}} V^{-\frac{c}{2\pi} J(p)} \quad \text{(disk)},$$

where $\ell > 0$ is fixed, and $\ln V \gg p$, and:

$$J(p) = p \ln\left(\frac{2}{n} \frac{p}{\sqrt{1 + p^2}}\right) + \operatorname{arccot}(p) - \arccos(n/2).$$

with $c = 1$ (**dilute**), $c = 1/[1 - \frac{1}{\pi} \arccos\left(\frac{n}{2}\right)]$ (**dense**), $c \in [1, 2]$, $n \in [0, 2]$.
See also *Borot, Garcia-Failde '16; Chen, Curien, Maillard '17; Budd '18*

Large Deviations Function



$J(p)$ for $n = 1$ (Ising & Percolation), $n = \sqrt{2}$ (FK Ising), $n = \sqrt{3}$ (3-state Potts), $n = 2$ (4-state Potts & CLE_4).

Conformal Loop Ensemble (CLE)

[Sheffield '09, Sheffield & Werner '12]

The critical $O(n)$-model on a regular planar lattice is predicted to converge in the continuum scaling limit to $\mathrm{SLE}_\kappa/\mathrm{CLE}_\kappa$, for

$$n = -2\cos\left(4\pi/\kappa\right), \qquad n \in (0,2], \qquad \begin{cases} \kappa \in (8/3, 4], & \text{dilute phase} \\ \kappa \in [4, 8), & \text{dense phase,} \end{cases}$$

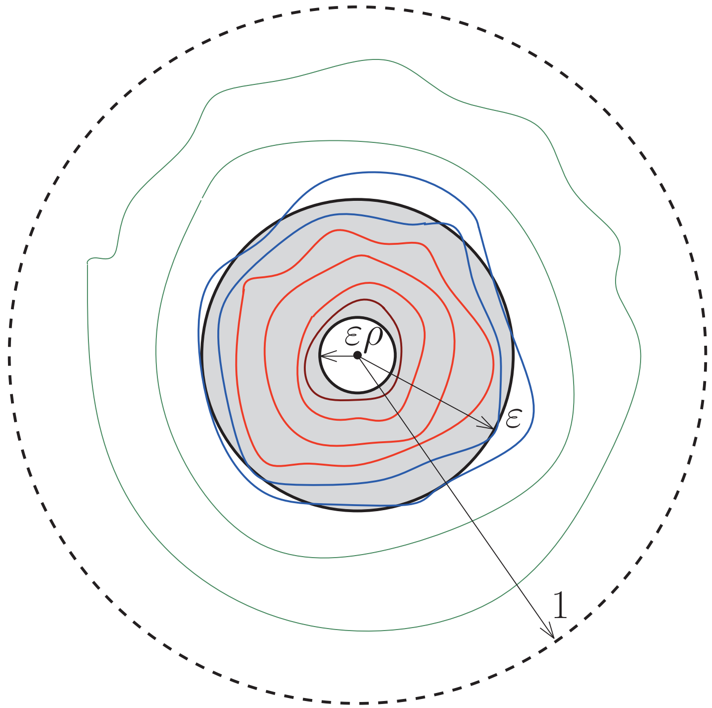
(Loop-erased random walk & spanning trees *[Lawler, Schramm, Werner]*, Ising & percolation *[Smirnov]*, GFF contour lines *[Schramm-Sheffield]*.)

On a **random planar map**: random measure conjectured to be, **after uniformization**, (a form of) the **Liouville quantum measure** μ_γ for

$$\gamma = \min\{\sqrt{\kappa}, 4/\sqrt{\kappa}\},$$

and **independent GFF & CLE** (*KPZ '88*, Q. Z., Q.I. *[Sheffield '10]*, M.o.T. *[D., Miller, Sheffield '14]*), $\gamma = \sqrt{8/3}$ *[Miller–Sheffield '15, '17]*)

Nesting in the Conformal Loop Ensemble (CLE)



$\mathcal{N}_z(\varepsilon)$ *is the* **number of nested loops** *of a* CLE_κ, $\kappa \in (8/3, 8)$ *surrounding the ball* $B(z, \varepsilon)$ *in the unit disk.*

Extreme nesting in CLE *[Miller, Watson & Wilson ’14]*

Let $\mathcal{N}_z(\varepsilon)$ be the number of loops of a CLE_κ, $\kappa \in (8/3, 8)$ surrounding the ball $B(z,\varepsilon)$, and Φ_ν the set of points z where

$$\lim_{\varepsilon\to 0} \mathcal{N}_z(\varepsilon)/\ln(1/\varepsilon) = \nu.$$

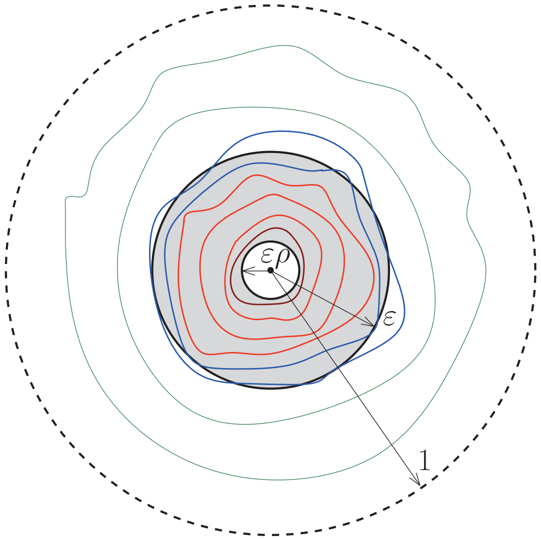
$$\dim_{\mathcal{H}} \Phi_\nu = 2 - \gamma_\kappa(\nu)$$

$$\gamma_\kappa(\nu) = \nu\Lambda_\kappa^*(1/\nu), \nu \geqslant 0; \quad \Lambda_\kappa^*(x) := \sup_{\lambda\in\mathbb{R}} (\lambda x - \Lambda_\kappa(\lambda))$$

$$\Lambda_\kappa(\lambda) = \ln\left(\frac{-\cos(4\pi/\kappa)}{\cos\left(\pi\sqrt{\left(1-\frac{4}{\kappa}\right)^2 + \frac{8\lambda}{\kappa}}\right)}\right)$$

Moment generating function of the loop log-conformal radius *[Cardy & Ziff ’02; Kenyon & Wilson ’04; Schramm, Sheffield & Wilson ’09]*

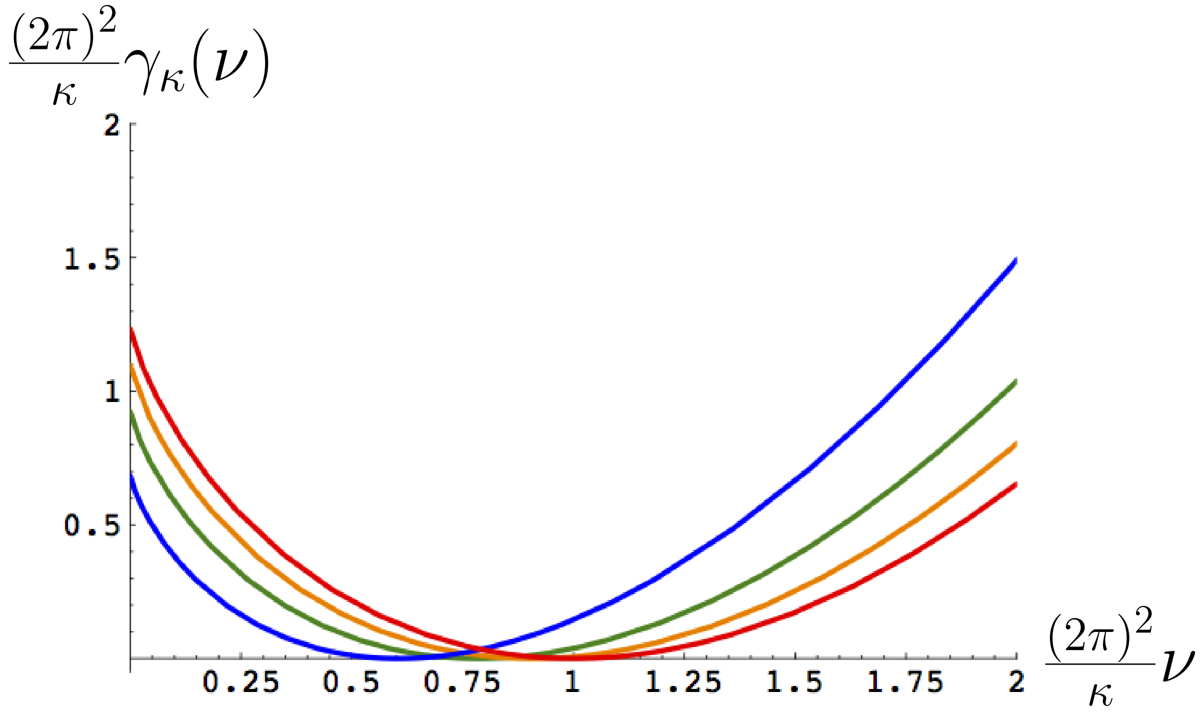
Conformal Loop Ensemble CLE_κ, $\kappa \in (8/3, 8)$



$\mathcal{U}$ the connected component containing 0 in the complement $\mathbb{D} \setminus \mathcal{L}$ of the largest loop $\mathcal{L}$ surrounding 0 in $\mathbb{D}$. Cumulant generating function of $T = -\ln(\mathrm{CR}(0, \mathcal{U}))$ *[Schramm, Sheffield, Wilson '09]*

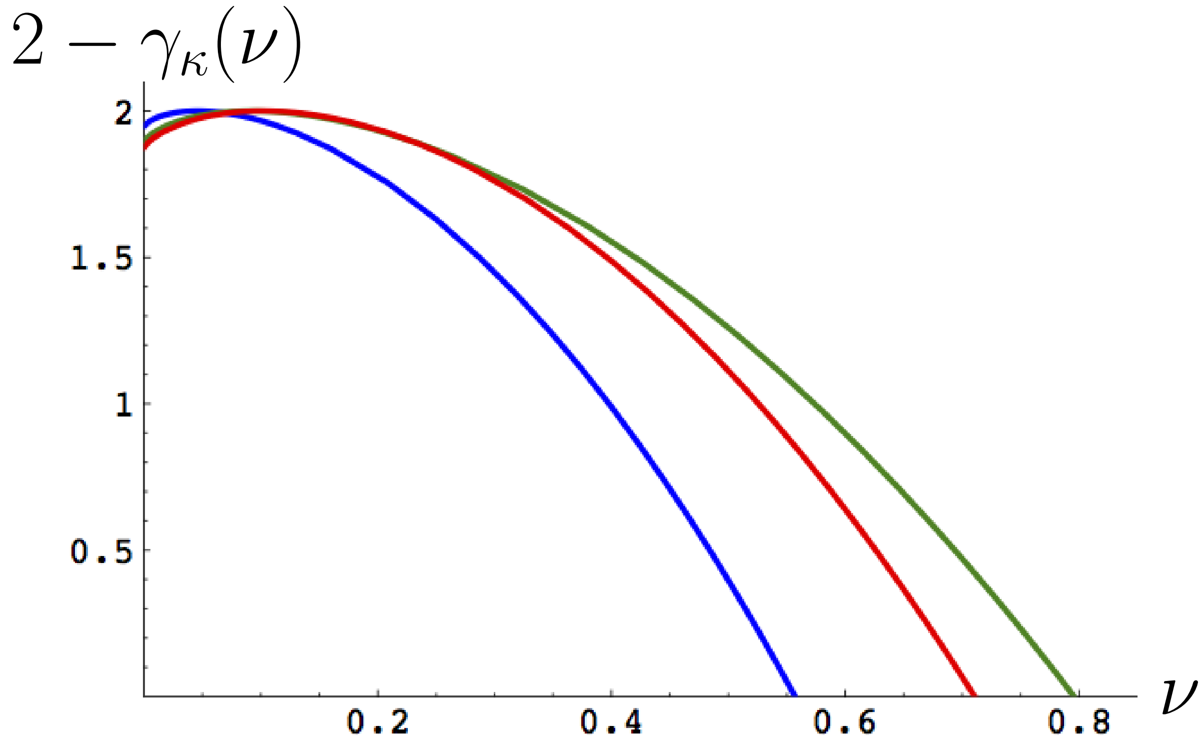
$$\Lambda_\kappa(\lambda) := \ln \mathbb{E}\left[e^{\lambda T}\right] = \ln\left(\frac{-\cos(4\pi/\kappa)}{\cos\left(\pi\left[\left(1-\frac{4}{\kappa}\right)^2 + \frac{8\lambda}{\kappa}\right]^{1/2}\right)}\right), \lambda \in \left(-\infty, 1 - \frac{2}{\kappa} - \frac{3\kappa}{32}\right).$$

Large Deviations Function



CLE$_\kappa$ **nesting large deviations function**, $\gamma_\kappa(\nu)/\kappa$,
for $\kappa = 3$ or 6 (Ising / Percolation, $n = 1$), $\kappa = 16/3$ (FK-Ising, $n = \sqrt{2}$),
$\kappa = 25/4$ (3-state Potts, $n = \sqrt{3}$), $\kappa = 4$ (GFF contour lines, $n = 2$)

Multifractal Spectrum



CLE$_\kappa$ **nesting Hausdorff dimension**, $\dim_{\mathcal{H}} \Phi_\nu = 2 - \gamma_\kappa(\nu)$,
for $\kappa = 3$ (Ising), $\kappa = 4$ (GFF contour lines), $\kappa = 6$ (Percolation).

Large Deviations

Euclidean case: for a ball of radius ε

$$\mathbb{P}\left(\mathcal{N}_z \approx \nu \ln(1/\varepsilon) \,\middle|\, \varepsilon\right) = \mathbb{P}\left(\mathcal{N}_z \approx \nu t \,\middle|\, t\right) \asymp \varepsilon^{\gamma_\kappa(\nu)} = \exp[-t\gamma_\kappa(\nu)].$$

Liouville Quantum Gravity:

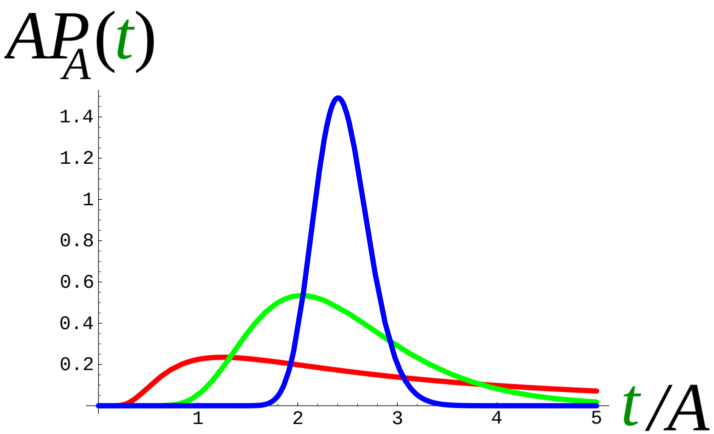
$$t := -\ln\varepsilon;\; A := -\gamma^{-1}\ln\delta, \quad \delta := \int_{B(z,\varepsilon)} \mu_\gamma \quad \text{(quantum ball)}$$

Conditioned on δ*, hence* A*, perform the convolution*

$$\mathbb{P}_Q(\mathcal{N}_z | A) := \int_0^\infty \mathbb{P}\left(\mathcal{N}_z \,|\, t\right) P(t|A)\,dt,$$

where $P(t|A)$ *is the probability distribution of the random Euclidean log-radius* t*,* **given** *the quantum log-radius* A*.*

Probability Distribution *[D.– Sheffield '09]*



$\gamma = \sqrt{8/3}$ $[A = 2;\ 20;\ 200]$

$$P(t\,|\,A) = \frac{A}{\sqrt{2\pi t^3}} \exp\left[-\frac{1}{2t}\left(A - a_\gamma t\right)^2\right]$$

$$t = -\ln\varepsilon,\ \ A = -\gamma^{-1}\ln\delta,\ \ \delta = \int_{B(z,\varepsilon)} \mu_\gamma$$

$$a_\gamma := 2/\gamma - \gamma/2$$

Quantum Large Deviations

$$t = -\ln\varepsilon,\ A = -\gamma^{-1}\log\delta \text{ (quantum ball)},$$

$$\mathcal{N} \approx -\nu\ln\varepsilon = \nu t,\ \mathcal{N} \approx -p\ln\delta = \gamma p A,$$

which implies $\nu t = \gamma p A$. The above convolution then yields, for $A \to +\infty$,

$$\mathbb{P}_Q(\mathcal{N}_z \approx \gamma p A \,|\, A) \asymp \int_0^\infty \frac{dt\,A}{\sqrt{2\pi t^3}} \exp\left(-\frac{(A-a_\gamma t)^2}{2t} - \gamma_\kappa(\nu)t\right)$$

$$\asymp \exp[-A\Theta(p)] \text{ (saddle point at constant } \nu\text{t)}$$

$\Theta(p)$ is the large deviations function for the loop number around a δ-quantum ball to scale as $p\log(1/\delta)$.

Legendre Transform & KPZ

In the plane, the Legendre transform gave

$$\gamma_\kappa(\nu) = \lambda - \nu \Lambda_\kappa(\lambda), \quad \frac{1}{\nu} = \frac{\partial \Lambda_\kappa(\lambda)}{\partial \lambda}.$$

In **Liouville Quantum Gravity**

$$\Theta(p) = U_\gamma^{-1}(\lambda/2) - p\Lambda_\kappa(\lambda), \quad \frac{1}{p} = \frac{\partial \Lambda_\kappa(\lambda)}{\partial U_\gamma^{-1}(\lambda/2)},$$

where $U_\gamma^{-1}(\lambda/2) := \left(\sqrt{a_\gamma^2 + 2\lambda} - a_\gamma\right)/\gamma$ is the inverse **KPZ** function, with

$$\gamma = \min\left\{\sqrt{\kappa}, 4/\sqrt{\kappa}\right\}, \quad a_\gamma = 2/\gamma - \gamma/2.$$

□

Theorem *[Borot, Bouttier, D. '16]*

In **Liouville quantum gravity**, *the cumulant generating function* Λ_κ, *with* $\kappa \in (8/3, 8)$, *is transformed into the quantum one,* $\Lambda_\kappa^{Q} := \Lambda_\kappa \circ 2U_\gamma$, *where* $U_\gamma(\lambda) := \left(1 - \frac{\gamma^2}{4}\right)\lambda + \frac{\gamma^2}{4}\lambda^2$ *is the KPZ function for* $\gamma = \min\{\sqrt{\kappa}, 4/\sqrt{\kappa}\}$. *Its Legendre-Fenchel transform is*

$$\Lambda_\kappa^{Q\star}(x) := \sup_{\lambda \in \mathbb{R}} \left(\lambda x - \Lambda_\kappa^{Q}(\lambda)\right).$$

The quantum nesting distribution in the disk is then, for $\delta \to 0$,

$$\mathbb{P}_Q(\mathcal{N}_z \approx p \ln(1/\delta) \,|\, \delta) \asymp \delta^{\Theta(p)},$$

$$\Theta(p) = \begin{cases} p\,\Lambda_\kappa^{Q\star}(1/p), & \text{if } p > 0 \\ 3/4 - 2/\kappa & \text{if } p = 0 \text{ and } \kappa \in (8/3, 4] \\ 1/2 - \kappa/16 & \text{if } p = 0 \text{ and } \kappa \in [4, 8). \end{cases}$$

Corollary *[Borot, Bouttier, D. '16]*

The **quantum** *generating function associated with* CLE_κ *nesting is, for* $\kappa \in (\frac{8}{3}, 8)$

$$\Lambda_\kappa^{Q}(\lambda) = \Lambda_\kappa \circ 2U_\gamma(\lambda) = \ln\left(\frac{\cos\left[\pi(1-4/\kappa)\right]}{\cos\left[\pi\left(2\lambda/c + |1-4/\kappa|\right)\right]}\right), \quad c = \max\{1, \kappa/4\},$$

$$\lambda \in \left[\tfrac{1}{2} - \tfrac{2}{\kappa}, \tfrac{3}{4} - \tfrac{2}{\kappa}\right] \text{ for } \kappa \in \left(\tfrac{8}{3}, 4\right]; \qquad \lambda \in \left[\tfrac{1}{2} - \tfrac{\kappa}{8}, \tfrac{1}{2} - \tfrac{\kappa}{16}\right] \text{ for } \kappa \in [4, 8).$$

- **The KPZ relation**, which usually concerns scaling dimensions, acts here on a conjugate variable in a Legendre transform.
- **The composition map** $\Lambda_\kappa \mapsto \Lambda_\kappa^Q = \Lambda_\kappa \circ 2U_\gamma$ to go from Euclidean geometry to Liouville quantum geometry is fairly general.

CLE Nesting in Liouville Quantum Gravity

Theorem *[Borot, Bouttier, D. '16]*

The **quantum** nesting probability of CLE_κ in a simply connected domain, for the number $\mathcal{N}_z$ of loops surrounding a ball centered at z and conditioned to have a given Liouville quantum measure δ, has the large deviations form,

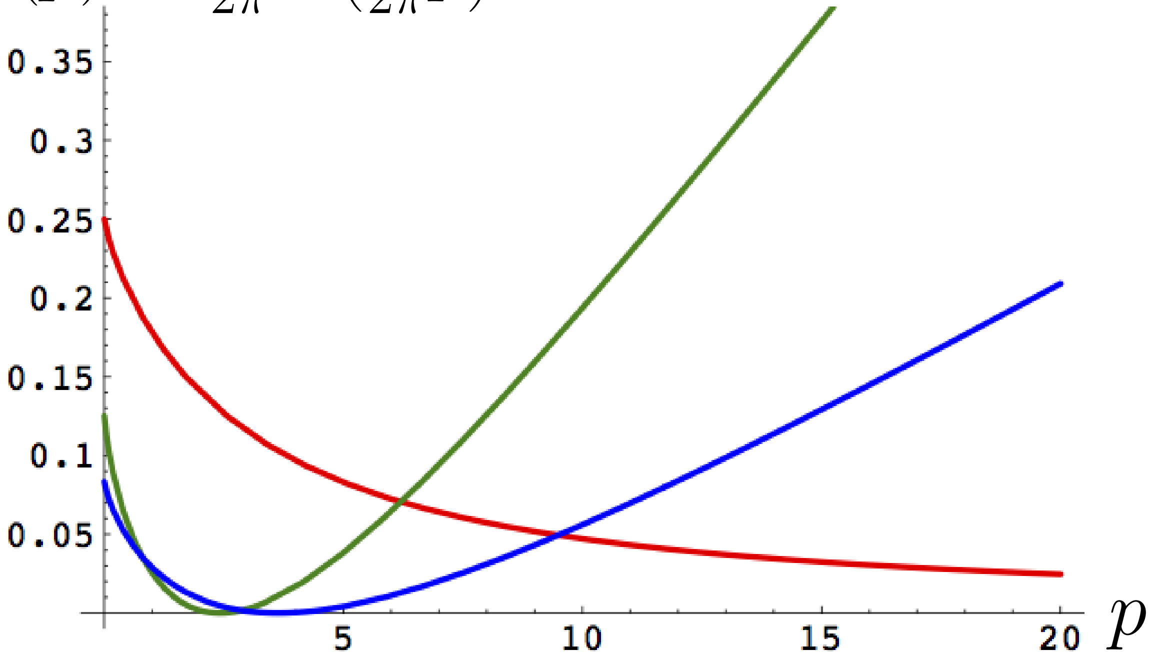
$$\mathbb{P}_Q\left(\mathcal{N}_z \approx \frac{cp}{2\pi}\ln(1/\delta)\,\middle|\,\delta\right) \asymp \delta^{\frac{c}{2\pi}J(p)}, \quad \delta \to 0,$$

$$\Theta\left(\frac{cp}{2\pi}\right) = \frac{c}{2\pi}J(p),$$

where c and J are the *same* as in the **combinatorial result** for the critical $O(n)$ model in the scaling limit of **large random maps**.

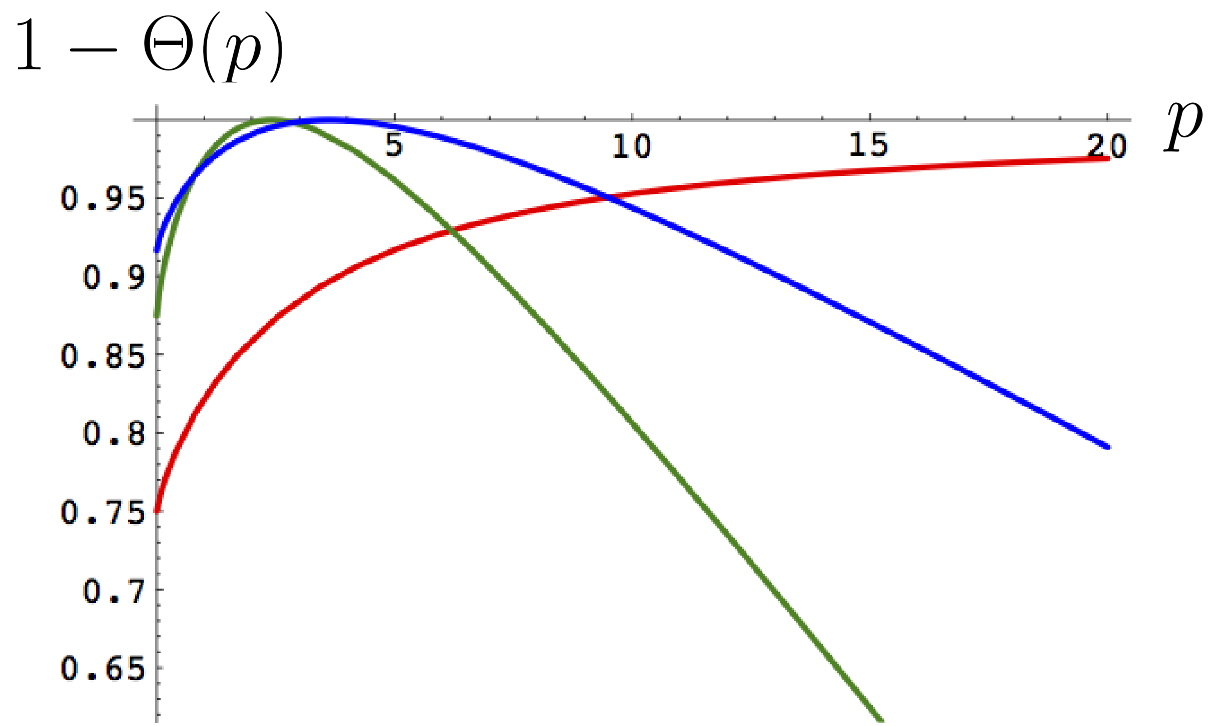
Quantum Large Deviations Function

$$\Theta(p) = \frac{c}{2\pi} J\left(\frac{c}{2\pi} p\right)$$



$\Theta(p)$ for $\kappa = 3$ (Ising), $\kappa = 4$ (GFF contour lines), $\kappa = 6$ (Percolation).

Quantum Multifractal Spectrum



$1-\Theta(p)$ for $\kappa = 3$ (Ising), $\kappa = 4$ (GFF contour lines), $\kappa = 6$ (Percolation).
Quantum Hausdorff Dimension for p-nesting points: $D_{\mathcal{H}}\left(1-\Theta(p)\right)$,
with $D_{\mathcal{H}}$ Hausdorff dimension of the **γ-Liouville quantum surface**.

CLE on the Riemann sphere *[Kemppainen & Werner '14]*

Theorem *[Borot, Bouttier, D. '16]*

The nesting probability in $\mathrm{CLE}_\kappa(\widehat{\mathbb{C}})$ *between two balls of radius* ε_1 *and* ε_2 *and centered at two distinct punctures, has the large deviations form,*

$$\mathbb{P}^{\widehat{\mathbb{C}}}\left[\mathcal{N}(\varepsilon_1,\varepsilon_2)\approx \nu \ln(1/(\varepsilon_1\varepsilon_2))\right] \asymp (\varepsilon_1\varepsilon_2)^{\gamma_\kappa(\nu)}, \quad \nu \geqslant 0, \quad \varepsilon_1,\varepsilon_2 \to 0,$$

where $\gamma_\kappa(\nu)$ *is the large deviations function of the disk topology.*

Corollary

For two balls of same radius ε,

$$\mathbb{P}^{\widehat{\mathbb{C}}}\left(\mathcal{N}(\varepsilon,\varepsilon)\approx \nu \ln(1/\varepsilon)\right) \asymp \varepsilon^{\widehat{\gamma}_\kappa(\nu)}, \quad \nu \geqslant 0, \quad \varepsilon \to 0,$$

where $\widehat{\gamma}_\kappa(\nu)$ *is related to the disk large deviations function by*

$$\widehat{\gamma}_\kappa(\nu) = 2\gamma_\kappa(\nu/2).$$

Quantum Riemann sphere

Theorem *[Borot, Bouttier, D. '16]*

On the quantum sphere $\widehat{\mathbb{C}}$, the large deviations function $\widehat{\Theta}$ which governs the nesting probability between two non-overlapping δ-quantum balls,

$$\mathbb{P}_{Q}^{\widehat{\mathbb{C}}}(\mathcal{N} \approx p \ln(1/\delta) \,|\, \delta) \asymp \delta^{\widehat{\Theta}(p)}, \qquad \delta \to 0,$$

*is related to the Θ function for the **disk** topology by*

$$\widehat{\Theta}(p) = 2\Theta(p/2),$$

so that

$$\mathbb{P}_{Q}^{\widehat{\mathbb{C}}}\left(\mathcal{N} \approx \frac{cp}{\pi} \ln(1/\delta) \,\Big|\, \delta \to 0\right) \asymp \delta^{\frac{c}{\pi} J(p)},$$

where c and J are the same as before.

Perfect matching of **LQG** results for CLE_κ with those for the $O(n)$ **model** on a **random planar map**, with the correspondence $\delta \leftrightarrow 1/V$, with $\delta \to 0$, $V \to +\infty$.