The Structure of the Spatial slices of 3-dimensional Causal Triangulations

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Outline

- The problem of counting triangulations
- Causal triangulations
- Description of the main result
- The midsection and its properties
- Bijection between causal slices and a class of 2-dimensional coloured cell complexes
- Possible extensions and interesting questions

Counting triangulations

- In two dimensions the problem was solved by Tutte (1962) and Bender and Canfield (1986)

$$N_{g,b}(n) \sim n^{5(g-1)/2+b-1}c^n$$

where $N_{g,b}(n)$ is the number of triangulations of a genus g surface with b boundary components made up of n triangles.

- No restriction on topology

$$N(n) = \sum_{g=0}^{\infty} N_{g,1}(n) \sim (3n/2)!$$

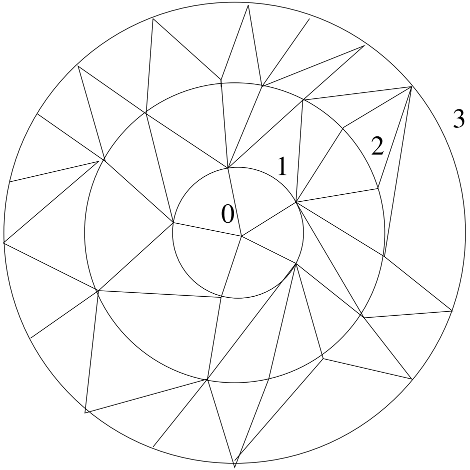
- Important for the analysis of partition functions for discrete quantum gravity in 2 dimensions.

3 dimensions

- Discrete models of 3-dimensional quantum gravity (Ambjørn, Durhuus, TJ 1991): Need bounds on the number of different triangulations of S^3 that can be constructed with a given number of tetrahedra.
- In order for
$$Z = \sum_{T \in \mathcal{T}} e^{-S_{EH}}$$
to converge for some κ where
$$S_{EH}(T) = \kappa |T| + \lambda \ell(T)$$
($|T|$ = number of tetrahedra in T, $\ell(T)$ = number of edges) we need
$$\#\{T \in \mathcal{T} : |T| = n\} \leq C^n \quad (*)$$
for some constant C.
- Not known whether the inequality $(*)$ holds.

Causal Triangulations

- Causal triangulations are simpler triangulations that are made up of a sequence of spatial slices (global hyperbolic structure) (Ambjørn, Jurkiewicz, Loll 2001)
- The inequality $(*)$ holds for causal triangulations in 3 dimensions (Durhuus and TJ 2015)
- **Main result:** There is a bijection between the spatial slices of 3-dimensional causal triangulations and a class of coloured 2-dimensional cell complexes that satisfy a number of conditions (work with B. Durhuus).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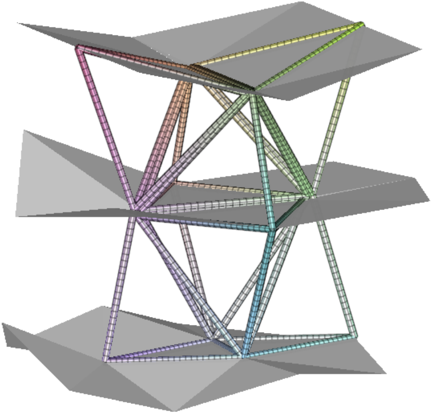
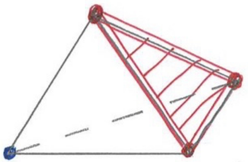


http://www.thephysicsmill.com/2013/10/13/causal-dynamical-triangulations/

3- dimensional Triangulations

- Building blocks: Tetrahedra with vertices coloured red or blue



- Not all of the same colour
- 3 types: (3,1), (2,2), (1,3)
- Can have monocoloured or two-coloured edges and triangles

Definition A *triangulation* K is a collection of tetrahedra some of whose sides (triangles) are pairwise identified, respecting the colouring

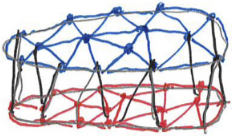
- The boundary of K, ∂K, is the set of all non identified triangles
- Regularity:
  (i) No two triangles in the same tetrahedron can be identified
  (ii) Two different triangles in a tetrahedron t cannot be identified with two triangles an a different tetrahedron t'
- Can view a triangulation:
  (a) as a topological space
  (b) a combinatorial object (abstract simplicial complex)
  (c) a subset of $\mathbb{R}^n$, n large enough, where each tetrahedron (triangle, edge) is the convex hull of its vertices (assumed to be affinely independent)

Definition A *causal disc-slice* is a triangulation K with the following properties

(i) K is homeomorphic to the 3-ball

(ii) All monocoloured simplices of K belong to the boundary ∂K such that the red ones form a disc D_r and the blue ones form a disc D_b

- $\partial K = D_r \cup D_b \cup C$ and C is a 2-dimensional causal slice



- There are no interior vertices
- There is a similar notion of causal sphere-slice which are homeomorphic to $S^2 \times [0, 1]$ and have two disjoint boundary components, one red and one blue

Definition A *causal disc-triangulation* is a triangulation of the form

$$M = \bigcup_{i=1}^{N} K_i$$

where K_i is a causal disc-slice with boundary discs D_r^i and D_b^i such that K_i and K_j are disjoint for $i \neq j$ except $D_b^i = D_r^{i+1}$, $i = 1, \ldots, N-1$, as uncoloured 2-dimensional triangulations.

- $\partial M = D_r^1 \cup D_b^N \cup C$

- Given two triangulated discs D_1 and D_2 there exists a causal disc slice K such that $D_r = D_1$ and $D_2 = D_r$.

The Midsection

- We can view any causal disc-slice K as imbedded in $\mathbb{R}^n$ ($n \geq 7$) such that each tetrahedron t is a convex linear combination of its vertices, i.e. $x \in t = (v_1 v_2 v_3 v_4)$, $v_j \in \mathbb{R}^n$, can be expressed as

$$x = \sum_{i=1}^{4} s_i v_i, \quad s_i \geq 0, \quad \sum_{i=1}^{4} s_i = 1$$

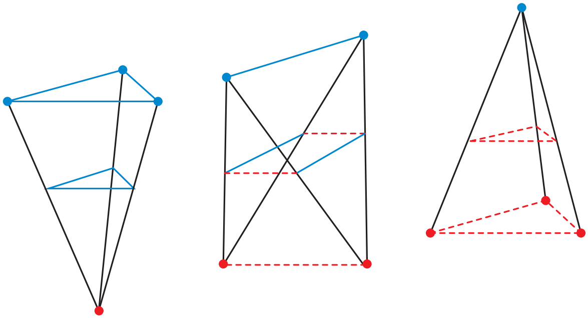
- Define a real valued function h on K

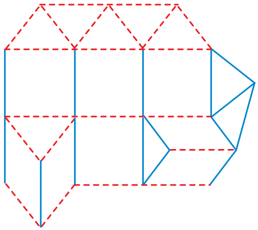
$$h(x) = \sum_{i : v_i \text{ red}} s_i \quad \text{(well defined)}$$

- The *midsection* of K is defined to be

$$S_K = \{x \in K : h(x) = 1/2\}$$

- The midsection S_K is made up of triangles with red edges or blue edges and two-coloured quadrangles with opposite edges of the same colour

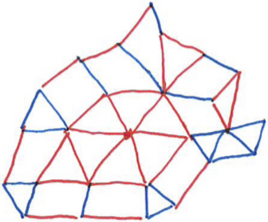




- If we contract the red edges in S_K we obtain D_b and contracting the blue edges yields D_r
- Edges, triangles, tetrahedra in K correspond to verticies, edges, 2-cells in S_K. We let e_a denote the edge in K which corresponds to the vertex a in S_K.
- S_K is a 2-dimensional cell complex (cells are triangles and quadrangles) with coloured edges and the topology of a disc
- Isomorphic causal disc-slices give rise to isomorphic midsections
- For sphere-slices the midsection is a 2-sphere

Properties of The Midsection

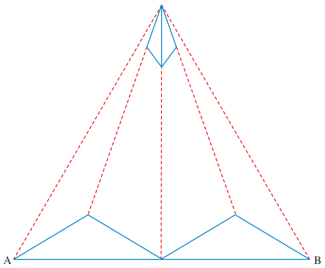
- We denote edges, triangles and quadrangles in the midsection by $\langle a_i a_j \rangle$, $\langle a_i a_j a_k \rangle$, $\langle a_i a_j a_k a_\ell \rangle$
- A red path in S_K is a sequence of red edges $\langle a_i a_{i+1} \rangle$, $i = 1, \ldots, k-1$. We say the path connects a_1 to a_k. It is simple if $a_i \neq a_j$, $i \neq j$ and we say it is closed if $a_1 = a_k$ and $a_i \neq a_j$, $i, j = 1, \ldots k-1$



Property α

Lemma 1 Two different vertices in S_K cannot be connected both by a red and by a blue path (property α)

Proof: If a and b are vertices in S_K connected by a blue path then the red endpoints of e_a and e_b are identical. If a and b are also connected by a red path then both the endpoints of e_a and e_b are the same so $e_a = e_b$.



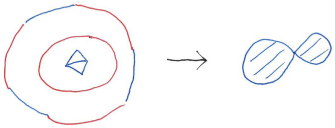
Not a midsection

Properties β_1 and β_2

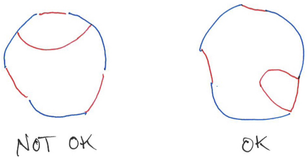
Lemma 2 (i) Let ρ be a closed red simple path in S_K. Then the interior of ρ contains only red edges (Property β_1)

(ii) Let μ be a simple red path connecting two vertices belonging to two different blue arcs of the boundary of S_K. Then the endpoints of μ are the endpoints of red boundary arc (Property β_2)

Proof of (i)



Proof of (ii)



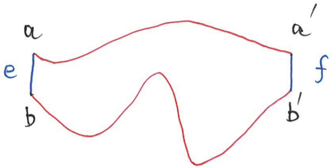
Property γ

- **Definition** Let e and f be different blue edges in S_K. We say they are connected by a blue path of quadrangles if

- **Lemma 3** Let $e = \langle ab \rangle$ and $f = \langle a'b' \rangle$ be different blue edges in S_K. Suppose a and a' as well as b and b' are connected by red paths. Then they are connected by a blue path of triangles. (Property γ)

Idea of proof

Let Δ_e and Δ_f be the two two-coloured triangles in K containing e and f. Then they share a blue edge (xy) in the blue boundary of K and they have red vertices v_e and v_f in ∂K, $v_e \neq v_f$. Looking at the "star" of (xy) in K, which contains a sequence of $(2,2)$ tetrahedra, we find the desired path of quadrangles.



The bijection

- Let $\mathcal{S}$ denote the set of all 2-dimensional cell complexes S with the topology of a disc

  (i) made up of red and blue triangles as well as two-coloured quadrangles with opposite sides of the same colour

  (ii) containing at least one triangle of each colour

  (iii) satisfying conditions α, β_1, β_2 and γ.

- Let $\mathcal{C}$ denote the set of all causal disc-slices.
- Define a mapping $\phi : \mathcal{C} \mapsto \mathcal{S}$ by $\phi(K) = S_K$.
- **Theorem** ϕ is a bijection.

Outline of proof

- Different disc-slices have different midsections so ϕ is injective.
- From any $S \in \mathcal{S}$ we construct a unique simplicial complex K_S. We show that this simplicial complex has the topology of a 3-ball and is in fact a disc-slice. The midsection of K_S is by construction the coloured cell complex S that we started with.
- To each a in the vertex set $V(S)$ of S we associate two (abstract) vertices r_a (red) and b_a (blue).
- Identify: $r_a = r_b$ if a and b are joined by a blue path in S and $b_a = b_b$ if a and b are joined by a red path.
- The vertex set $\{r_a, b_a : a \in V(S)\}$ (with the identifications described above) is the vertex set K_S^0 of an abstract simplicial complex K_S.

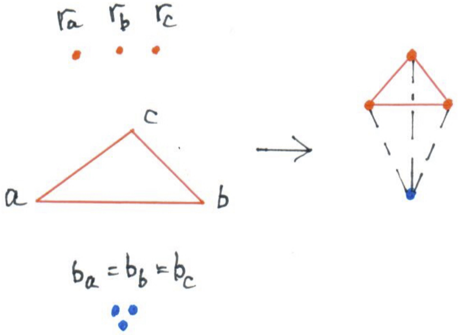
- The set of 3-simplicies K_S^3 is obtained from the 2-cells of S

$$\text{red triangle } \triangle = \langle abc \rangle \mapsto t_\triangle = (r_a r_b r_c b_a)$$

$$\text{blue triangle } \triangle = \langle abc \rangle \mapsto t_\triangle = (b_a b_b b_c r_a)$$

$$\text{quadrangle } \square = \langle abcd \rangle \mapsto t_\square = (r_a r_b b_a b_c)$$

- This is well defined by condition α and defines a 3-dimensional simplicial complex K_S whose 3-simplicies (tetrahedra) are labelled by the 2-cells of S



- Two tetrahedra t_F and $t_{F'}$ share a triangle if and only if the 2-cells F and F' share an edge
- The monocoloured triangles of K_S are labelled by the triangles of S and the two-coloured triangles of K_S are labelled by the edges of S
- The monocoloured edges in K_S lie in the boundary and the two-coloured edges are labelled by the vertices of S
- All monocoloured simplicies are in the boundary ∂K_S
- There is a one to one correspondence between the boundary edges of S and the two-coloured triangles in ∂K_S and these triangles form a 2-dimensional causal slice
- **Lemma** K_S has the topology of a 3-ball so ∂K_S is a 2-sphere
- It follows that K_S is a sphere slice with midsection S

Locally constructible simplicial manifolds

A 3d simplicial manifold M has a local construction if there is a sequence of simplicial manifolds $M_1, M_2, \ldots, M_k$ such that

(i) M_1 is a tetrahedron

(ii) M_{i+1} is obtained from M_i by either gluing a tetrahedron to M_i along a triangle or by identifying two triangles in ∂M_i which already share an edge

(iii) $M_k = M$

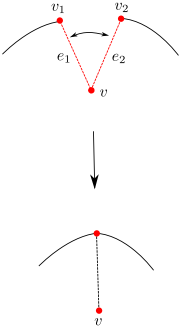
There is an analogous notion of local construction for 2 and higher-dimensional simplicial manifolds

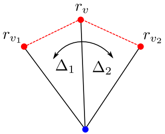
Facts about LC triangulations

- Any 2-dimensional simplicial manifold with the topology of S^2 or the 2-disc has a LC
- There is a $C > 1$ such that the number of locally constructible triangulations of S^3 of volume V is bounded by C^V (Durhuus and TJ 1995)
- Not all triangulations of S^3 are locally constructible (Benedetti and Ziegler 2011)

Outline of the proof of the Lemma

- Take a local construction of S
- Use it to obtain an alternative construction of K_S: $K_1, K_2, \ldots, K_n = K_S$
- K_1 is a single tetrahedron
- K_j is obtained from K_{j-1} by gluing a single tetrahedron along a triangle to ∂K_{j-1} or by identifying two triangles in ∂K_{j-1} which share an edge

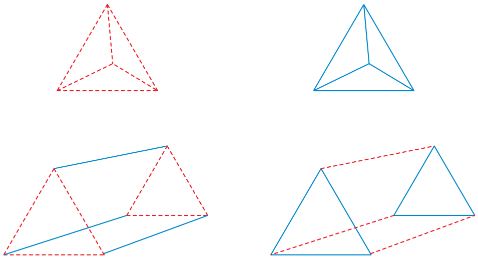




- K_1 is a 3-ball and the topology does not change as we go from K_{j-1} to K_j so K_S is a 3-ball
- This proves the Lemma and the Main Result
- All the results generalise to the case of sphere-slices with minor modifications

Generalisation to 4 dimensions

- One can generalise the definition of a causal triangulation to any dimension.
- One can generalise the construction of a midsection to 4-dimensional causal slices.
- There are 4 types of 4-simplicies that arise: (1,4), (2,3), (3,2) and (4,1).
- The midsection is a 3-dimensional cell complex made up of coloured tetrahedra and prisms:



- The midsection is in this case a 3-dimensional cell complex
- These cell complexes are not well understood. In particular, we do not have an exponential bound on their number as a function of the number of 3-cells
- However, if we have an exponential bound on the number of midsections that arise then we obtain an exponential bound on the number of causal 4-dimensional triangulations as a function of the number of 4-simplicies

Final Remarks

- Bijections between labelled trees and 2-dimensional triangulations have been an important tool in the study of 2-dimensional triangulations in recent years. The bijection we have described here is the first generalisation to 3 dimensions
- Is there a bijection between triangulations of S^3 and some labelled "2-dimensional structures"?
- There is still work to be done on causal triangulations in 3 dimensions: What midsections arise in a causal slice with given the boundary discs?
- Can we count the number causal slices (with given boundary discs) exactly or get asymptotic results about their number?
- Transfer matrix for 3-dimensional causal triangulations?