The Laplacian on some round Sierpiński carpets and Weyl's asymptotics for its eigenvalues

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1 Different geometries of the Sierpiński carpet

(generalized) self-similar SCs

Barlow–Bass '89, '99: Constr./Analysis of "B.M." Construction & Analysis of "Laplacian" & "B.M."?

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Dirichlet form & B.M. on self-similar SCs • A self-similar regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ exists.

(Barlow–Bass '89, '99, Kusuoka–Zhou '92)



BB '89: $\exists 1 \tau > 1$, $\{ Law(\{B_{\tau^n t}^{ref, D_n}\}_{t \ge 0})\}_{n=0}^{\infty}$ is tight.

• Such a regular Dirichlet form (\mathcal{E}, \mathcal{F}) is unique. (Barlow–Bass–Kumagai–Teplyaev '10)

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4/12 2 Some Kleinian groups G_m with $\partial_{\infty}G_m$ a RSC $> m > 6 \left(\frac{\pi}{2} + \frac{\pi}{3} + \frac{\pi}{m} < \pi \right)$ $\triangleright \{\ell_k\}_{k=1}^3$: \mathbb{B}^2 -geodesics, form \triangle , angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{m}$ $arprop \Gamma_{oldsymbol{m}} := ig\langle \{ \mathbf{Inv}_{oldsymbol{\ell}_{oldsymbol{k}}} \}_{oldsymbol{k}=1}^{oldsymbol{3}}ig angle$ $\sim \mathbb{B}^2 = \bigcup_{\tau \in \Gamma_m} \tau(\triangle_{\ell_1 \ell_2 \ell_3})$ $= G = G_m := \langle \Gamma_m, \operatorname{Inv}_S \rangle$ $\sim \partial_{\infty}G_m$ is a round SC.

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3 Results for Apollonian gasket: $K_{\alpha,\beta,\gamma_{embedding}}$ \square Thm(K., *cf.* Teplyaev '04). $\exists^1(\mathcal{E}^{\alpha,\beta,\gamma},\mathcal{F}^{\alpha,\beta,\gamma})$: str. local, irreducible, regular symmetric Dirichlet form over $K_{lpha,eta,\gamma}$, h_x, h_y are $\mathcal{E}^{\alpha,\beta,\gamma}$ -harmonic on $K_{\alpha,\beta,\gamma} \setminus V_0!$ Rmk. Choice of a reference measure is irrelevant: $\alpha \in \alpha$ $\mathcal{C}^{\alpha,\beta,\gamma} := \mathcal{F}^{\alpha,\beta,\gamma} \cap C(K_{\alpha,\beta,\gamma})$ and $\mathcal{E}^{\alpha,\beta,\gamma}|_{\mathcal{C}^{\alpha,\beta,\gamma}}$ are unique.

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3 Results for Apollonian gasket: $K_{\alpha,\beta,\gamma_{embedding}}$ Thm(K., *cf.* Teplyaev '04). $\exists 1(\mathcal{E}^{\alpha,\beta,\gamma}, \mathcal{F}^{\alpha,\beta,\gamma})$: str. local, irreducible, regular symmetric Dirichlet form over $K_{lpha,eta,\gamma}$, h_x, h_y are $\mathcal{E}^{\alpha, \beta, \gamma}$ -harmonic on $K_{\alpha, \beta, \gamma} \setminus V_0!$ **Rmk.** Choice of a reference measure is irrelevant: $\mathcal{C}^{\alpha,\beta,\gamma} := \mathcal{F}^{\alpha,\beta,\gamma} \cap C(K_{\alpha,\beta,\gamma})$ and $\mathcal{E}^{\alpha,\beta,\gamma}|_{\mathcal{C}^{\alpha,\beta,\gamma}}$ are unique. Thm(K.). LIP|_{$K_{\alpha,\beta,\gamma}$} is a core of $(\mathcal{E}^{\alpha,\beta,\gamma}, \mathcal{F}^{\alpha,\beta,\gamma})$, and $\forall u \in \text{LIP}, \ \mathcal{E}^{\alpha,\beta,\gamma}(u,u) = \sum_{C \subset \operatorname{arc} K_{\alpha,\beta,\gamma}} \operatorname{rad}(C) \int_C |\nabla_C u|^2 d\operatorname{vol}_C.$ $\triangleright \mu^{\alpha,\beta,\gamma} := \sum_{C \subset \operatorname{arc} K_{\alpha,\beta,\gamma}} \operatorname{rad}(C) d\operatorname{vol}_C : \operatorname{volume meas.}_{(\operatorname{NOT doubling}!)}$

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4 Laplacian on the limit set $\partial_{\infty}G$ of $G = G_m$

7/12



 $cf. \, \mathcal{E}^{\alpha,\beta,\gamma}(u,u) = \sum_{C \subset \operatorname{arc} K_{\alpha,\beta,\gamma}} \operatorname{rad}(C) \int_C |\nabla_C u|^2 \, d\operatorname{vol}_C,$ $\mu^{\alpha,\beta,\gamma} = \sum_{C \subset \operatorname{arc} K_{\alpha,\beta,\gamma}} \operatorname{rad}(C) \, d\operatorname{vol}_C.$

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7/12



 $\triangleright \mathcal{G} := \{g \in \mathsf{M\"ob}(\widehat{\mathbb{C}}) \mid g^{-1}(\infty) \in \widehat{\mathbb{C}} \setminus \overline{\mathbb{B}^2} \}$ $\triangleright K_0 := \mathbb{B}^2 \cap \partial_{\infty} G, \ K_g := g(K_0) \begin{pmatrix} g \in \mathcal{G} \text{ represents} \\ \mathsf{choice of initial} \, \Delta \end{pmatrix}$

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$\triangleright \nu^{g} := \sum_{C \subset \operatorname{arc} K_{g}} \operatorname{rad}(C) \operatorname{dvol}_{C} (\operatorname{NOT doubling!})$ $\triangleright^{\forall} u \in \operatorname{LIP}_{K_{g}}, \, \mathcal{E}^{g}(u, u) := \sum_{C \subset \operatorname{arc} K_{g}} \operatorname{rad}(C) \int_{C} |\nabla_{C} u|^{2} \operatorname{dvol}_{C} (cf. \operatorname{Osada} \operatorname{'07})$

Prop. On $L^2(K_g, \nu^g)$, $(\mathcal{E}^g, \operatorname{LIP}_c(K_g))$ is closable & its closure $(\mathcal{E}^g, \mathcal{F}_g)$ is a strongly local regular Dirichlet form.

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9/12 $\triangleright g \in \mathcal{G}$ (represents choice of the initial \mathbb{B}^2 - \triangle) **Prop.** $\Delta_{(K_a,\nu^g,\mathcal{E}^g,\mathcal{F}_a)}$ has discrete spectrum. $\triangleright \{\lambda_n^{g,U}\}_{n \in \mathbb{N}}$: the eigenvalues of $-\Delta_{(U,\nu^g,\mathcal{E}^g,\mathcal{F}_g)}$ $\triangleright \mathcal{N}_{g,U}(\lambda) := \#\{n \in \mathbb{N} \mid \lambda_n^{g,U} \leq \lambda\} \ (\emptyset \neq U_{\subset K_q} \text{ open})$

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 $\mathcal{H}^{d}(\partial_{K_{g}}U) = 0 \Rightarrow \lim_{\lambda \to \infty} \lambda^{-d/2} \mathcal{N}_{g,U}(\lambda) = c_{m} \mathcal{H}^{d}(U).$

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Rmk. $\sum_{n} e^{-t\lambda_{n}^{g,U}} = \int_{U} p_{t}^{g,U}(x,x) d\nu^{g}(x) \stackrel{t\downarrow 0}{\sim} c\mathcal{H}^{d}(U) t^{-d/2}$

⇔ Thm, BUT

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 $\begin{array}{l} \mathsf{Rmk.} \sum_{n} e^{-t\lambda_{n}^{g,U}} = \int_{U} p_{t}^{g,U}(x,x) d\nu^{g}(x) \overset{t\downarrow 0}{\sim} c\mathcal{H}^{d}(U) t^{-d/2} \\ \Leftrightarrow \mathsf{Thm, BUT} \ p_{t}^{g,U}(x,x) \asymp_{c_{x}} t^{-1/2} \ \mathsf{for} \ \nu^{g} \mathsf{-a.e.} \ x \in U! \end{array}$













- A "self-similar" decomp. ("fund. dom." for $G \cap \partial_{\infty} G$) (requires concrete knowledge of $G \cap \partial_{\infty} G$; NOT extend easily)
- A version of the "2-dimensional" Nash inequality (\rightsquigarrow the spectrum of Δ_{K_g} is discrete & $\exists p_t^{K_g}(x,y) \leq c_g t^{-1}$) • $\iota: K_g \hookrightarrow \mathbb{R}^2$ is \mathcal{E}^g -harmonic & $\Gamma_{\mathcal{E}^g}(\iota, \iota) = \nu^g$ ($\rightsquigarrow \{\langle X^g, z \rangle\}_t$ slower than $\{B_t^{\mathbb{R}}\}_t \rightsquigarrow \mathbb{P}_q[\tau_{\mathcal{D}}(z, v) \leq t]$ small)

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- •(*cf.* Bonk '11) The circles in $\partial_{\infty}G$ are unif. rel. separated: $\forall j \neq k, \operatorname{dist}(C_j, C_k) \geq \delta_m \min\{\operatorname{rad}(C_j), \operatorname{rad}(C_k)\}.$

6 Open Problem: OTHER round SCs?

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- RSCs obtained as quasi-sym. images of s.-s. SCs?

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(generalized) self-similar SCs

Bonk '11: Each of them can be quasi-symmetrically mapped to a round SC in a unique way!

 $\begin{array}{l} \text{Thm}(\mathsf{K}.). \ ^{\exists}c_{0} \in (0,\infty), \ ^{\forall}\alpha,\beta,\gamma \in (0,\infty),\\ \lim_{\lambda \to \infty} \#\{n \in \mathbb{N} \mid \lambda_{n}^{\alpha,\beta,\gamma} \leq \lambda\}/\lambda^{d/2} = c_{0} \mathcal{H}^{d}(K_{\alpha,\beta,\gamma}). \end{array}$

Prf. To follow Kigami–Lapidus' method [CMP '93], we use Kesten's renewal thm for Markov chains [Ann. Prob. '74].

 $\triangleright K_x \setminus V_0 = \bigcup_{k=1}^6 \bigcup_{l=1}^\infty K_{\phi_{k,l}(x)}$ $\triangleright \Gamma := \{ x_{=(\alpha,\beta,\gamma)} | \mathcal{H}^d(K_x) = 1 \}$ (the space of "Euc. shapes" of AGs)

 $arproptop \{ [X_n] \}_{n=0}^{\infty}$: Markov chain on Γ $x \sim [\phi_{k,l}(x)]$ w.prob. $\mathcal{H}^d(K_{\phi_{k,l}(x)})$ (note $\sum_{k,l} \mathcal{H}^d(K_{\phi_{k,l}(x)}) = 1$) $arproptop V_n := -rac{1}{d} \log \mathcal{H}^d(K_{X_n})$

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p. 30, Figure 3 of R. D. Mauldin & M. Urbański, Adv. Math. 136 (1998), 26–38

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Prf. To follow Kigami–Lapidus' method [CMP '93], we use Kesten's renewal thm for Markov chains [Ann. Prob. '74].

 $\triangleright K_x \setminus V_0 = \bigcup_{k=1}^6 \bigcup_{l=1}^\infty K_{\phi_{k,l}(x)}$ $\triangleright \Gamma := \{ x_{=(\alpha,\beta,\gamma)} | \mathcal{H}^d(K_x) = 1 \}$ (the space of "Euc. shapes" of AGs)

 $igsquare \{ [X_n] \}_{n=0}^{\infty}$: Markov chain on Γ , $x \sim [\phi_{k,l}(x)]$ w.prob. $\mathcal{H}^d(K_{\phi_{k,l}(x)})$ (note $\sum_{k,l} \mathcal{H}^d(K_{\phi_{k,l}(x)}) = 1$)

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 $= \mathbb{E}_{x} \left[\sum_{n=0}^{\infty} \mathcal{R}([X_{n}], s - V_{n}) \right]$ $\stackrel{s \to \infty}{\underset{\text{Kesten 74}}{\overset{s \to \infty}{\underset{\text{$

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15/12 A key for Reminder estimate: **Embedding** in H^1 ! $\triangleright \nu^{g} := \sum_{C \subset \operatorname{arc} K_{g}} \operatorname{rad}(C) \operatorname{dvol}_{C} (\operatorname{NOT doubling!})$ $\triangleright^{\forall} u \in \operatorname{LIP}|_{K_{g}}, \, \mathcal{E}^{g}(u, u) := \sum_{C \subset \operatorname{arc} K_{g}} \operatorname{rad}(C) \int_{C} |\nabla_{C} u|^{2} \operatorname{dvol}_{C} (cf. \operatorname{Osada'07})$

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ight)\!\int_{0}^{lpha}\!\!f(re^{i heta})\,rac{d heta}{lpha}+rac{s}{r}f(re^{i heta}).$ $\triangleright \nu^g := \sum_{C \subset \operatorname{arc} K_q} \operatorname{rad}(C) \operatorname{dvol}_C$ (NOT doubling!) $\triangleright^{\forall} u \in \mathrm{LIP}|_{K_g}, \, \mathcal{E}^g(u, u) := \sum_{C \subset \mathrm{arc} K_g} \mathrm{rad}(C) \int_C |\nabla_C u|^2 \, d\mathrm{vol}_C \, (cf. \, \mathrm{Osada} \, '\mathrm{O7})$

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