The Laplacian on some round Sierpiński carpets and Weyl's asymptotics for its eigenvalues

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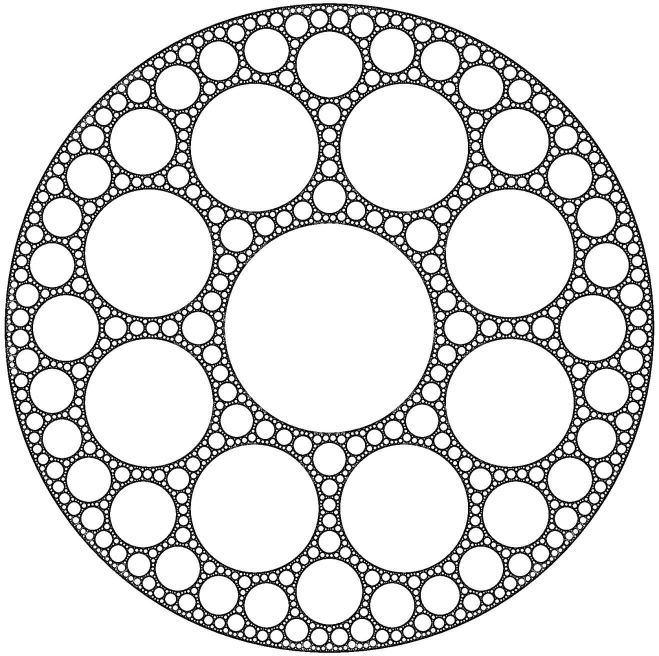
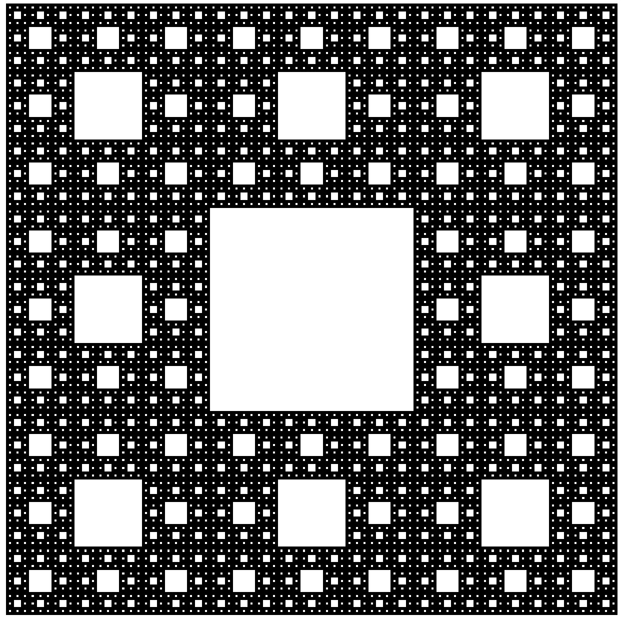
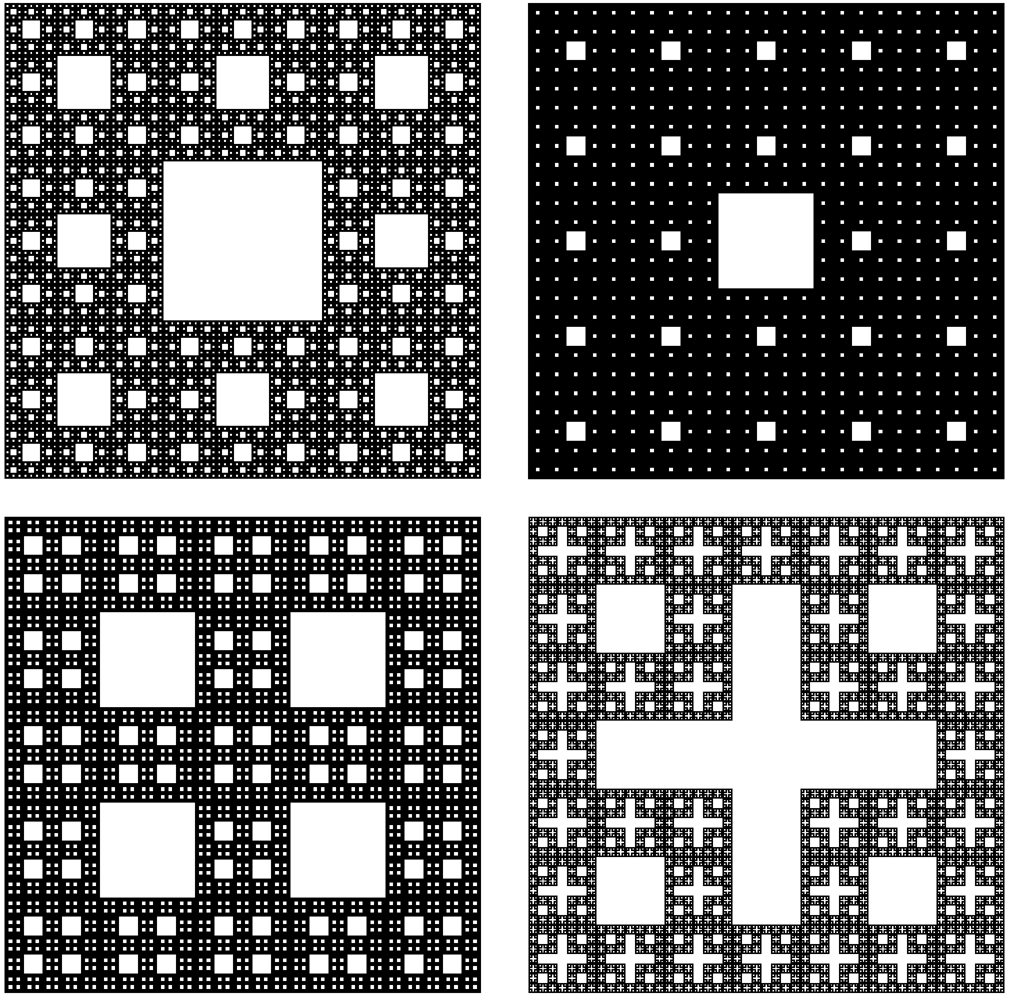


November 7, 2018
18:00–19:00



1 Different geometries of the Sierpiński carpet



(generalized) self-similar SCs

(Barlow–Bass '89, '99:
Constr./Analysis of "B.M.")

Construction & Analysis of
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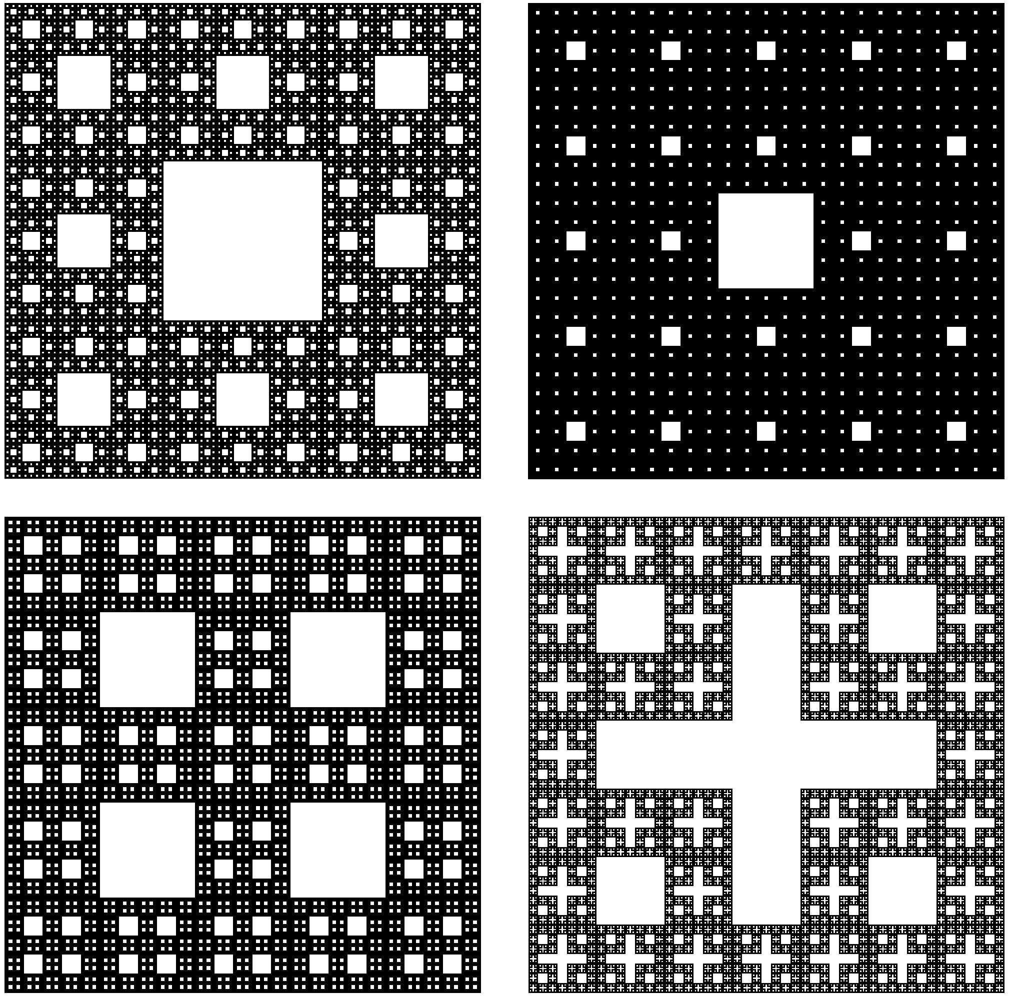

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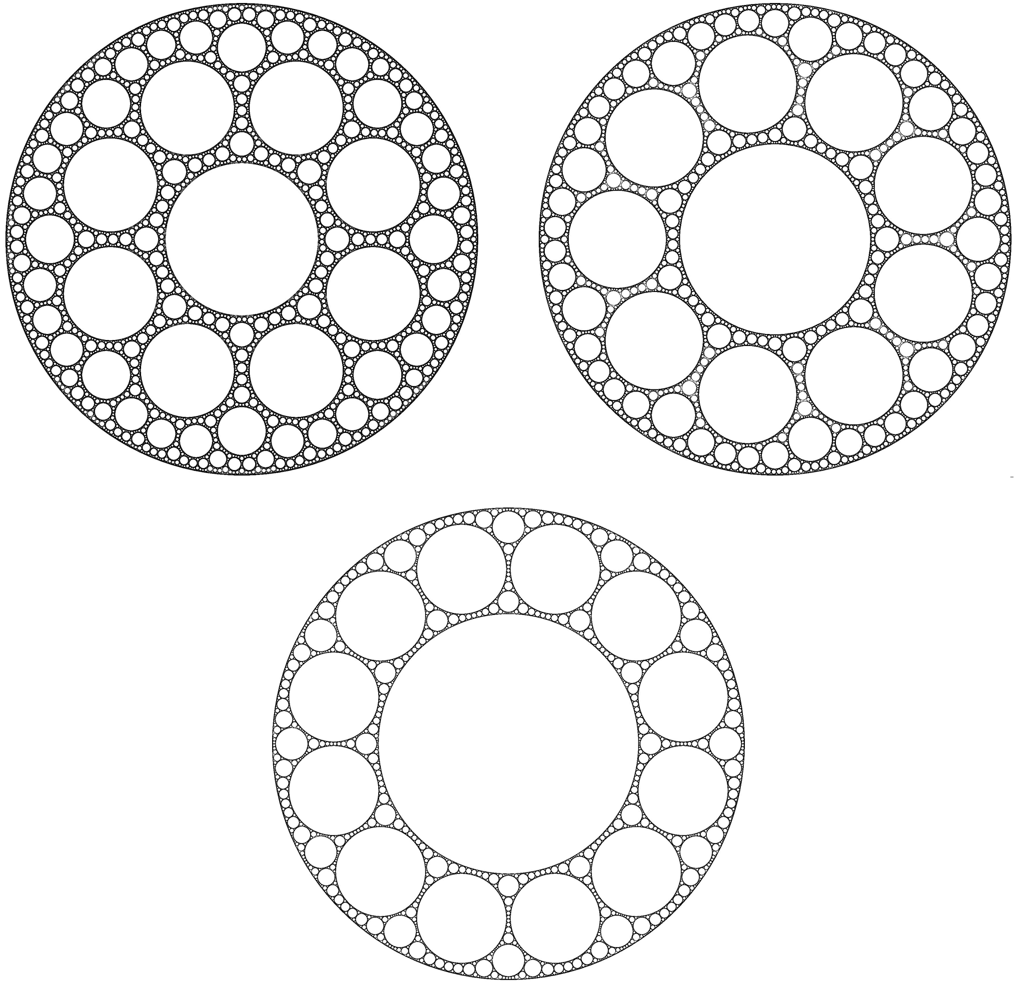

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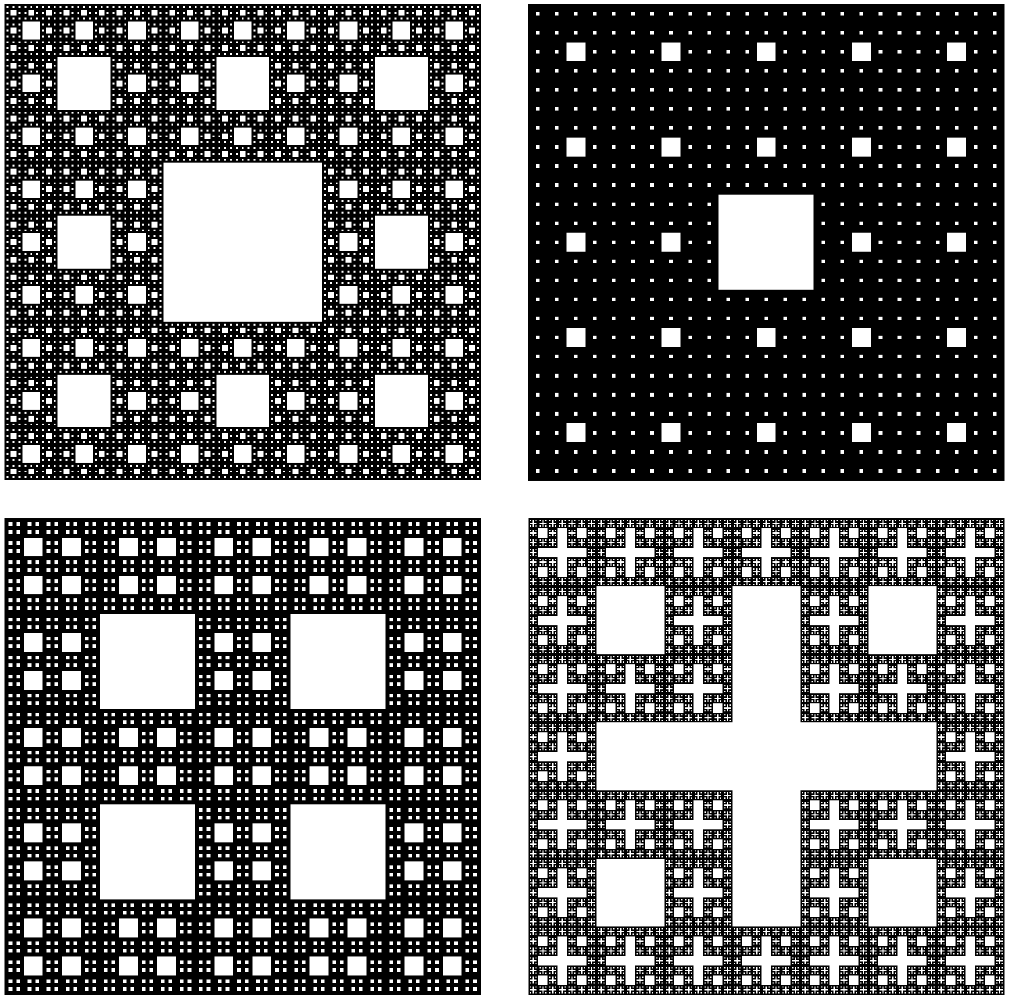

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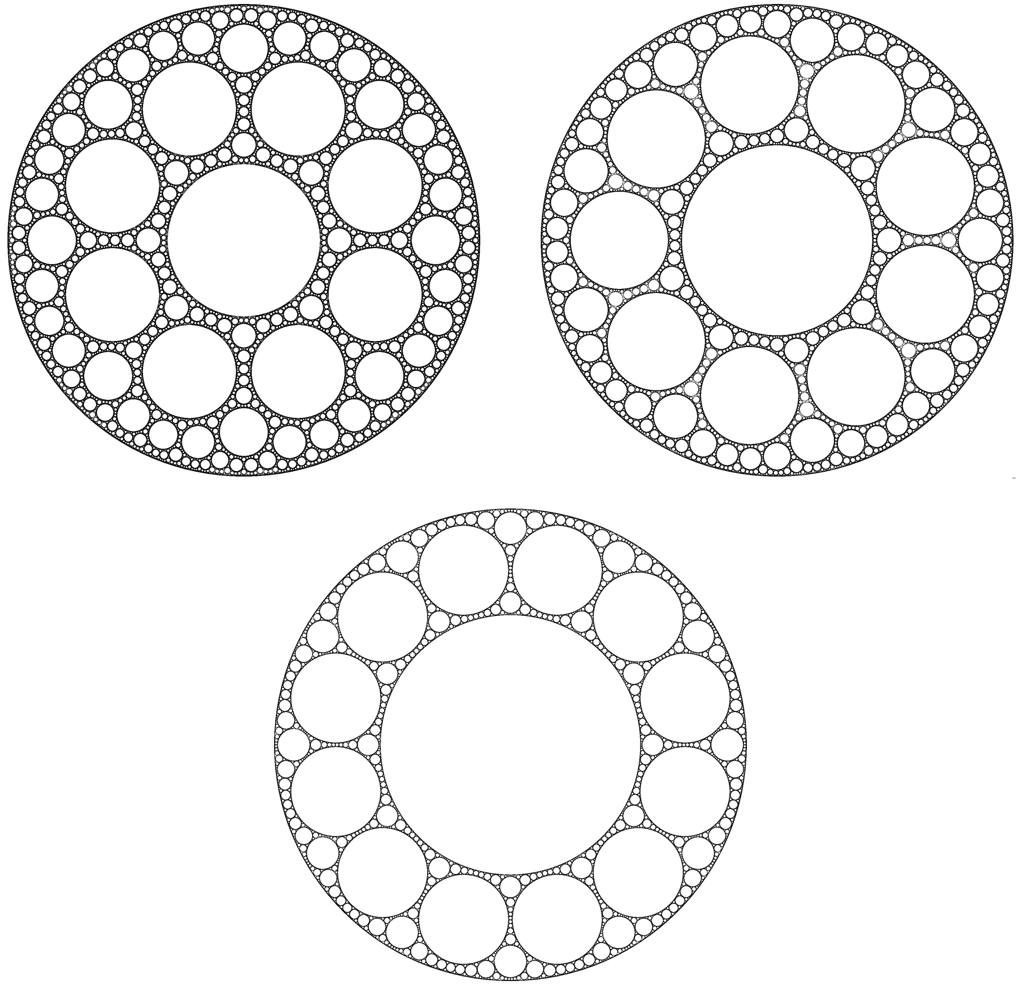

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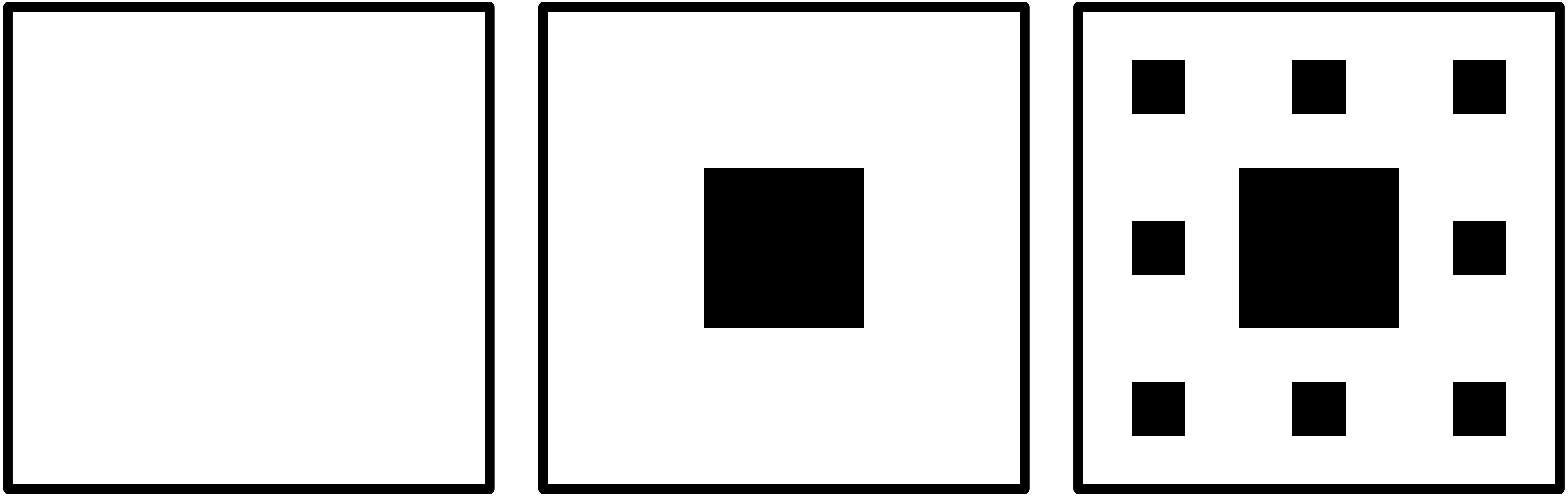
Dirichlet form & B.M. on self-similar SCs

- A self-similar regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ exists.

(Barlow–Bass '89, '99, Kusuoka–Zhou '92)



BB '89: $^{\exists 1}\tau > 1, \{\mathrm{Law}(\{B^{\mathrm{ref}, D_n}_{\tau^n t}\}_{t \geq 0})\}_{n=0}^{\infty}$ is tight.

- Such a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ is unique.

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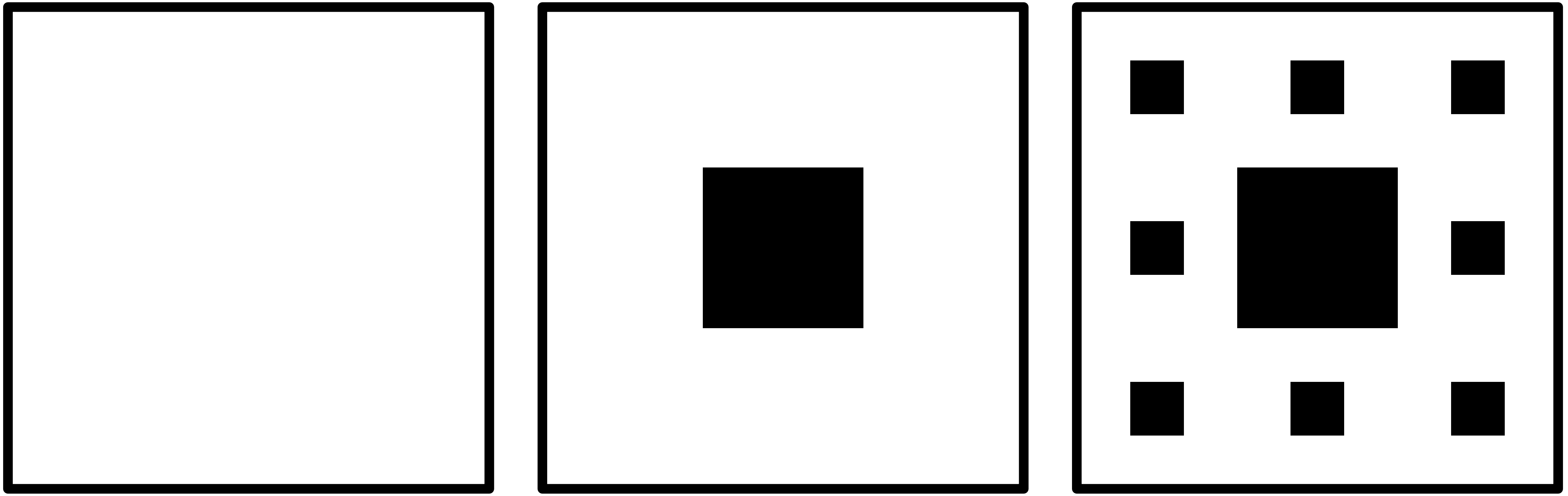

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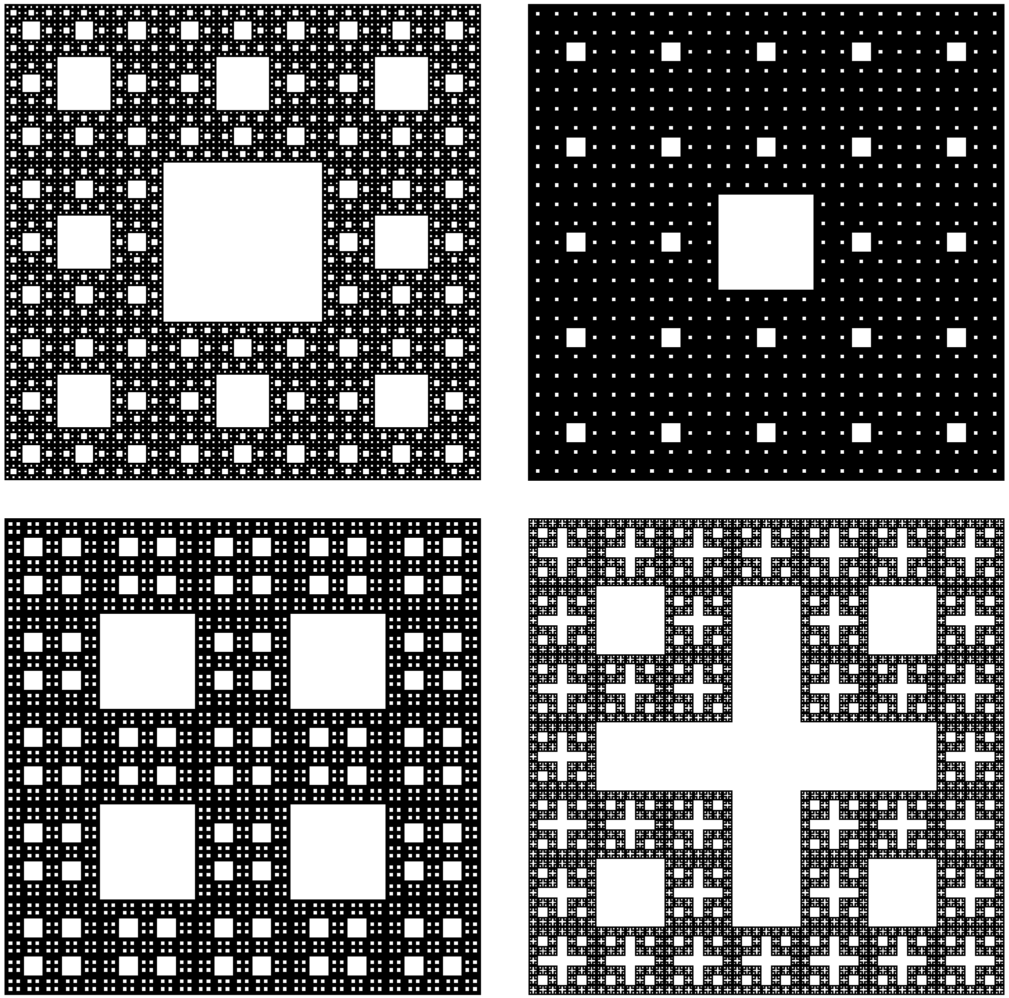

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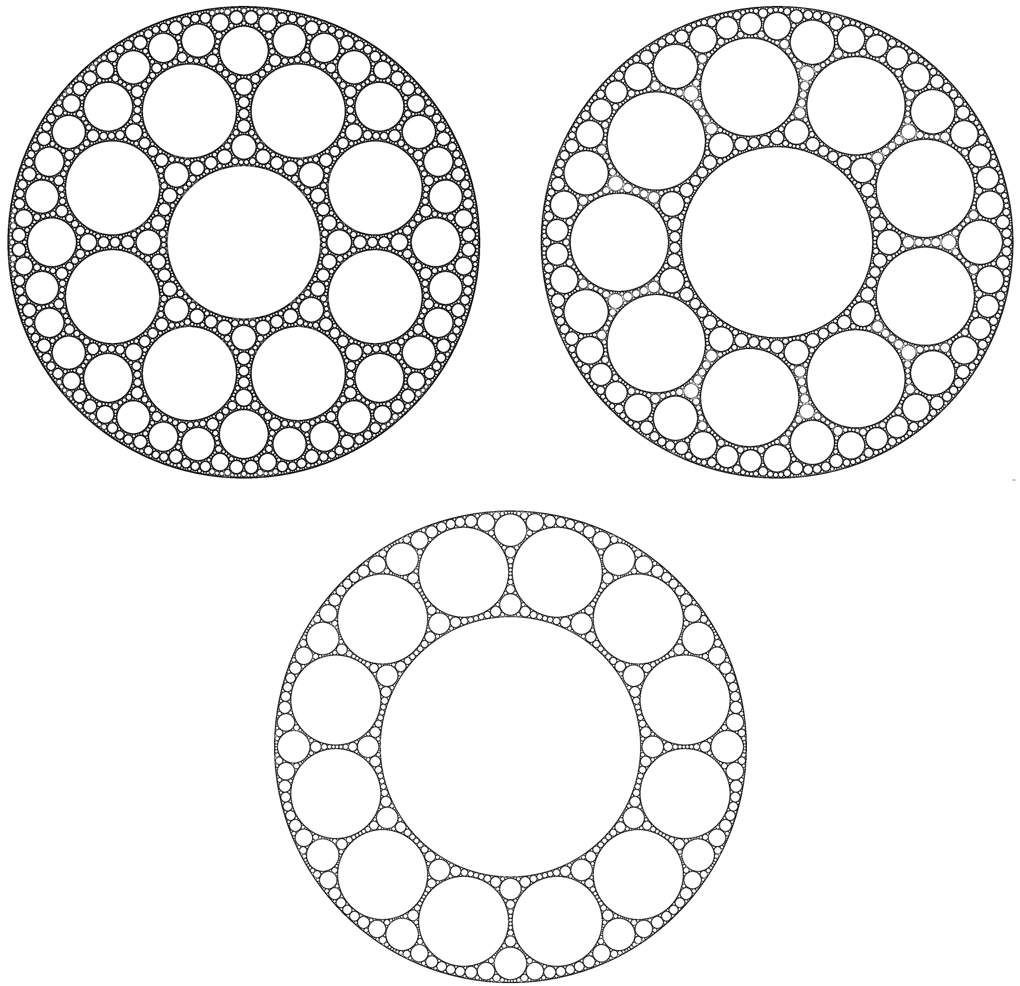

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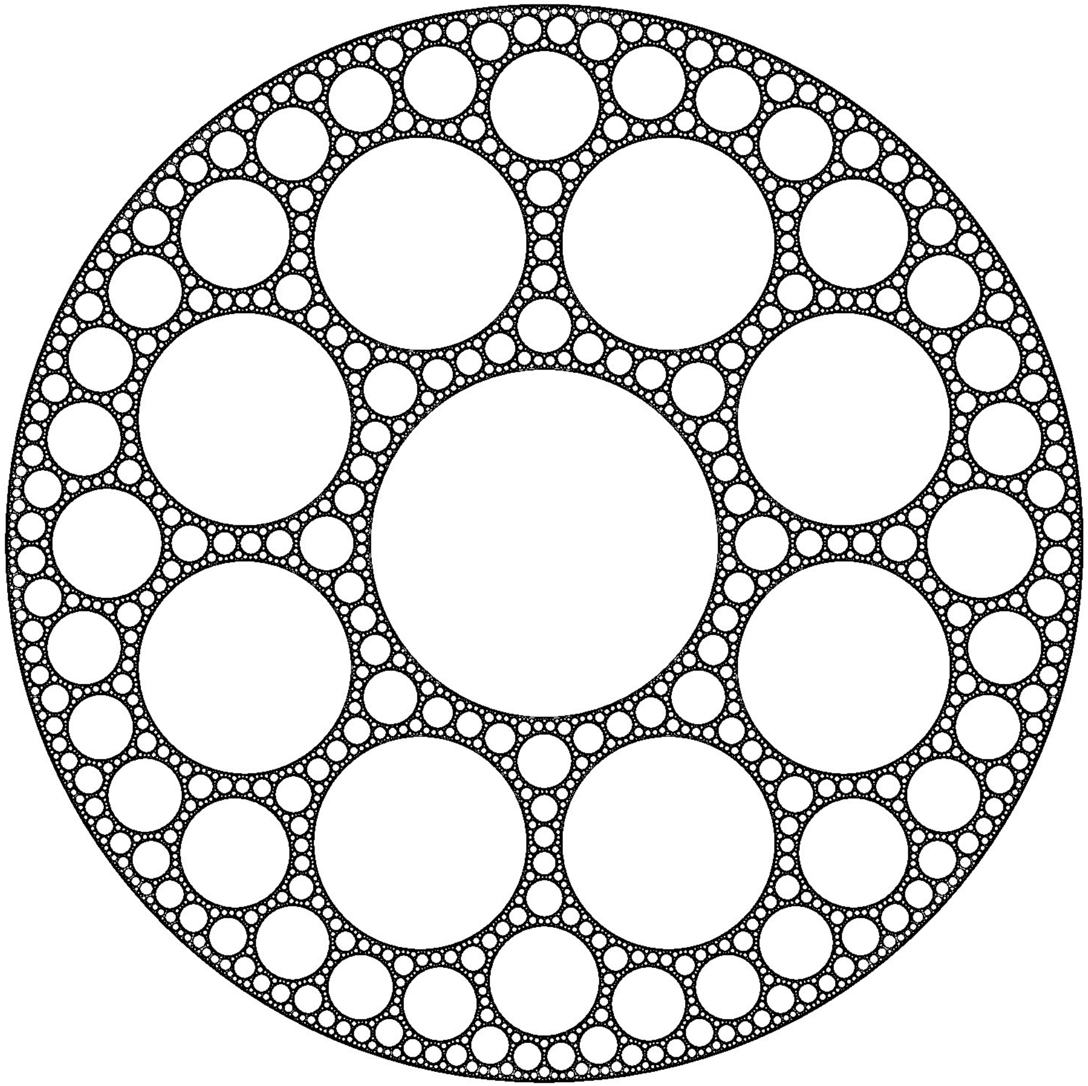

▷ $m > 6$ $\left(\frac{\pi}{2}+\frac{\pi}{3}+\frac{\pi}{m}<\pi\right)$

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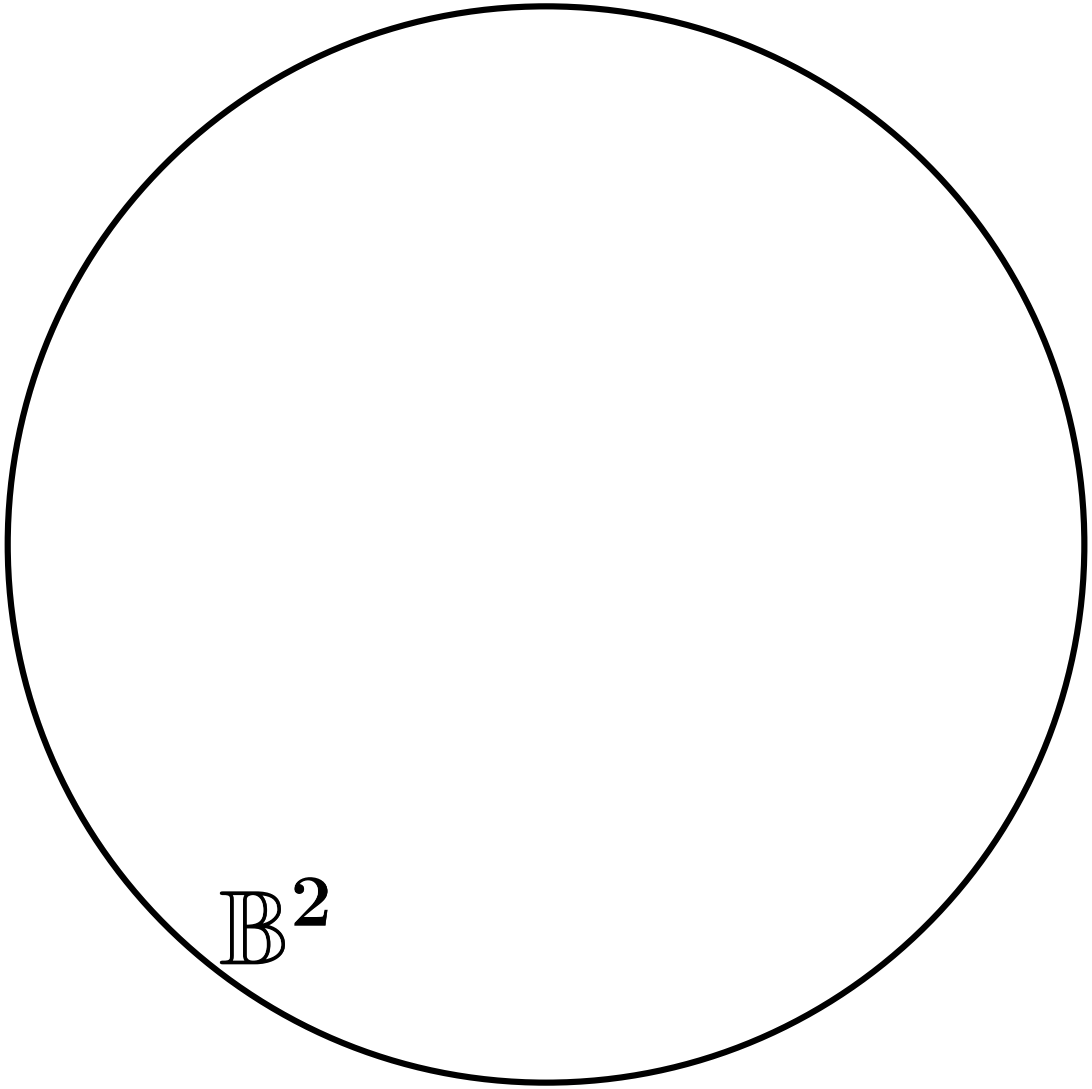

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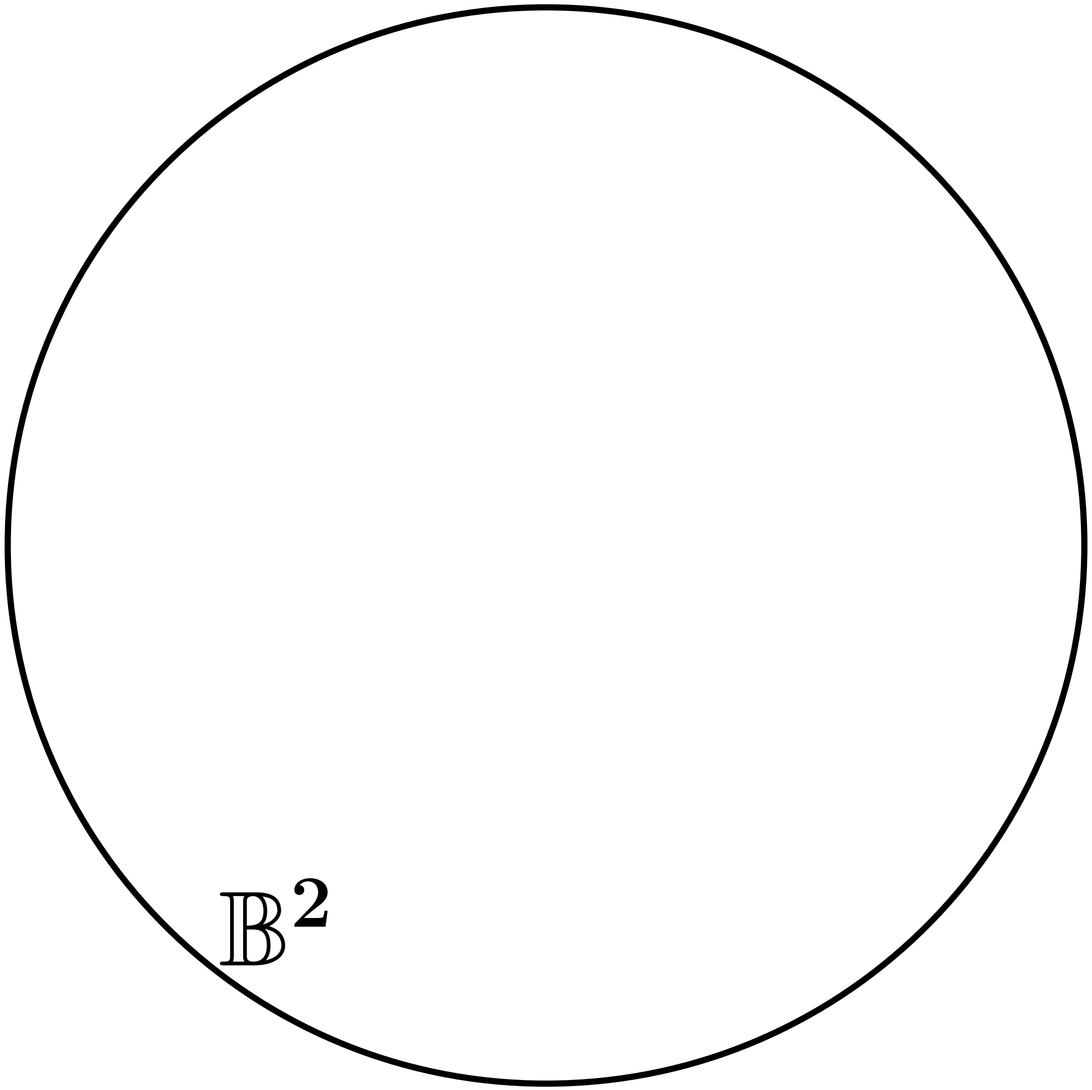

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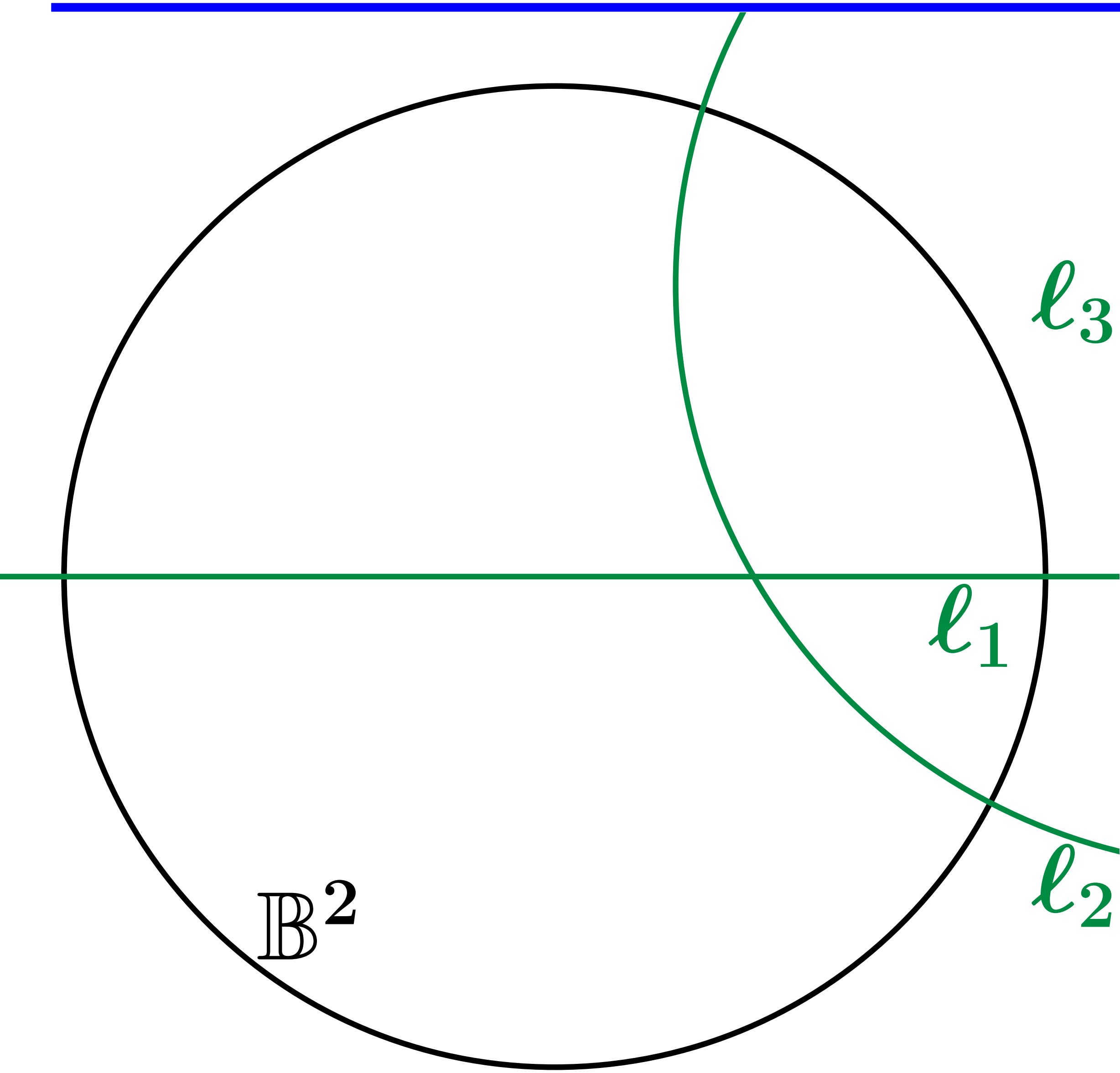

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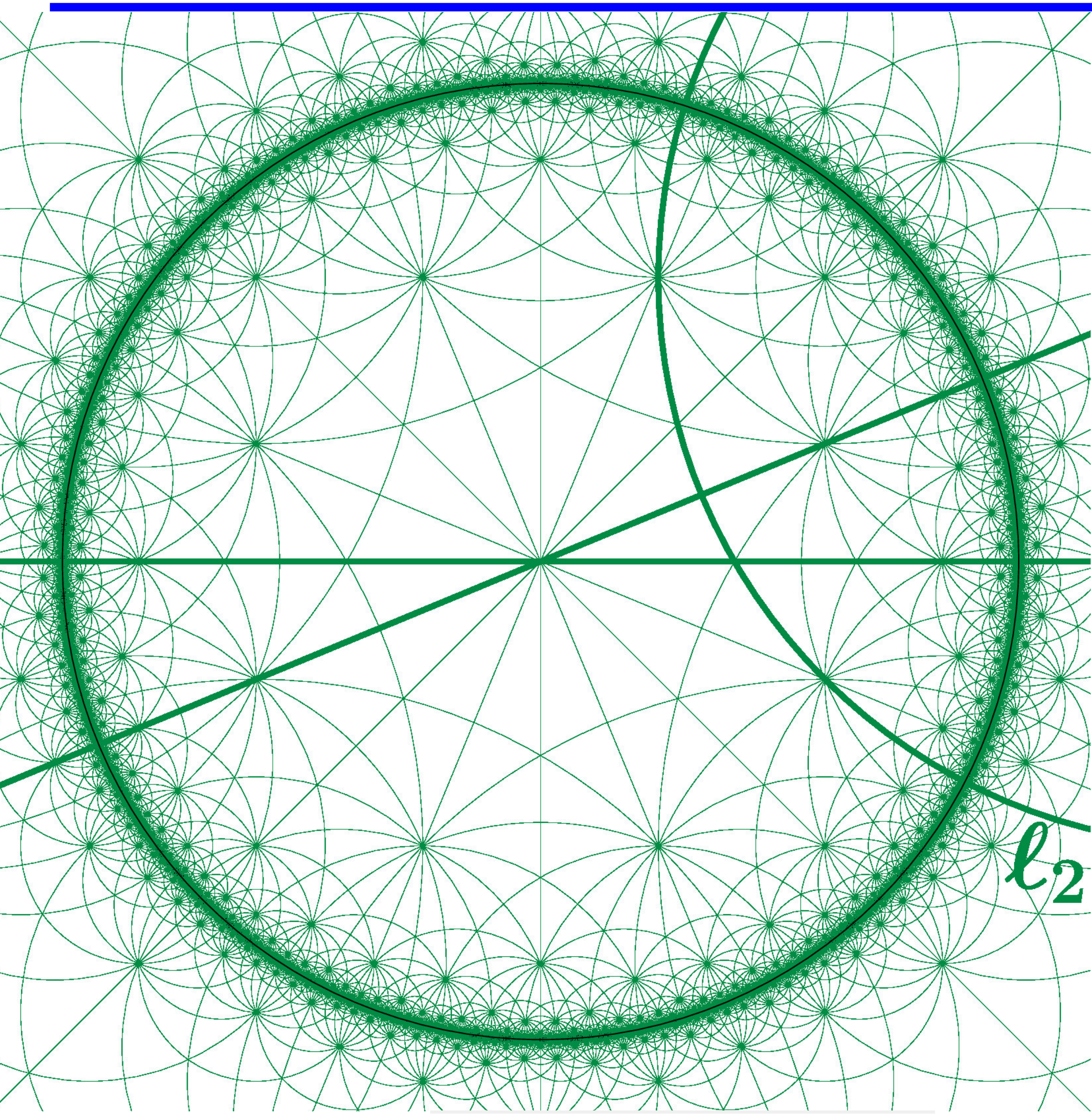

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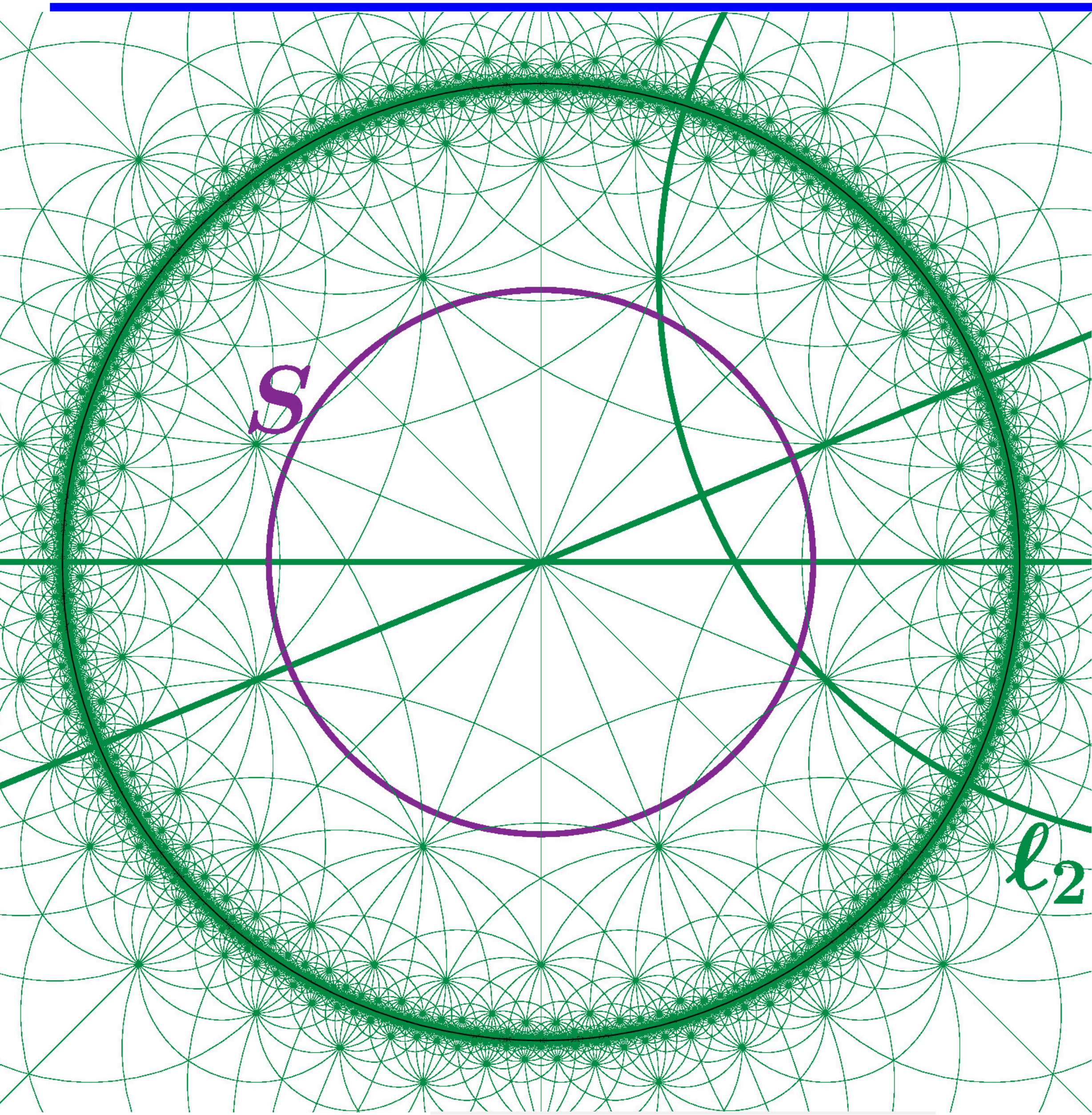

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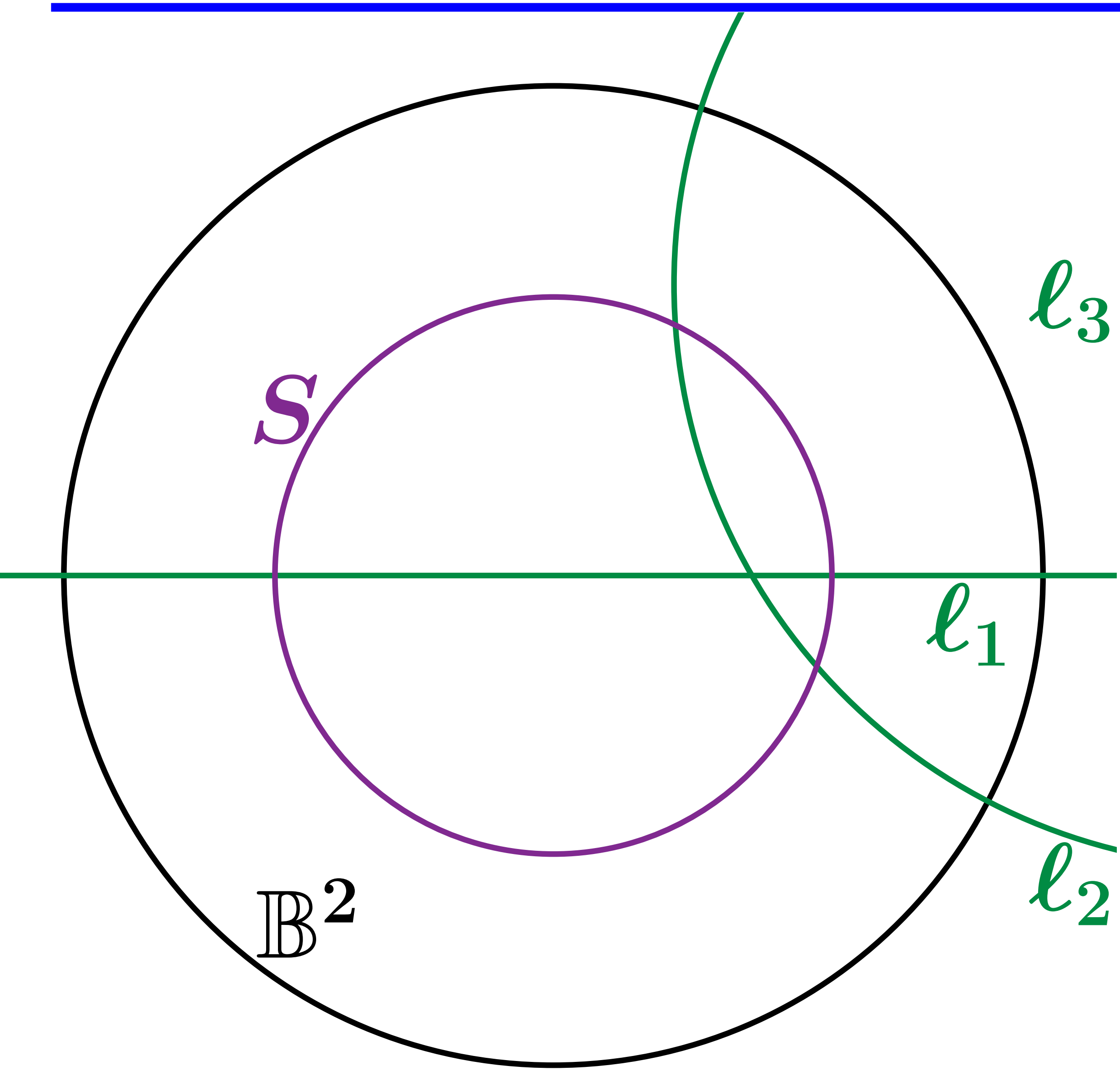

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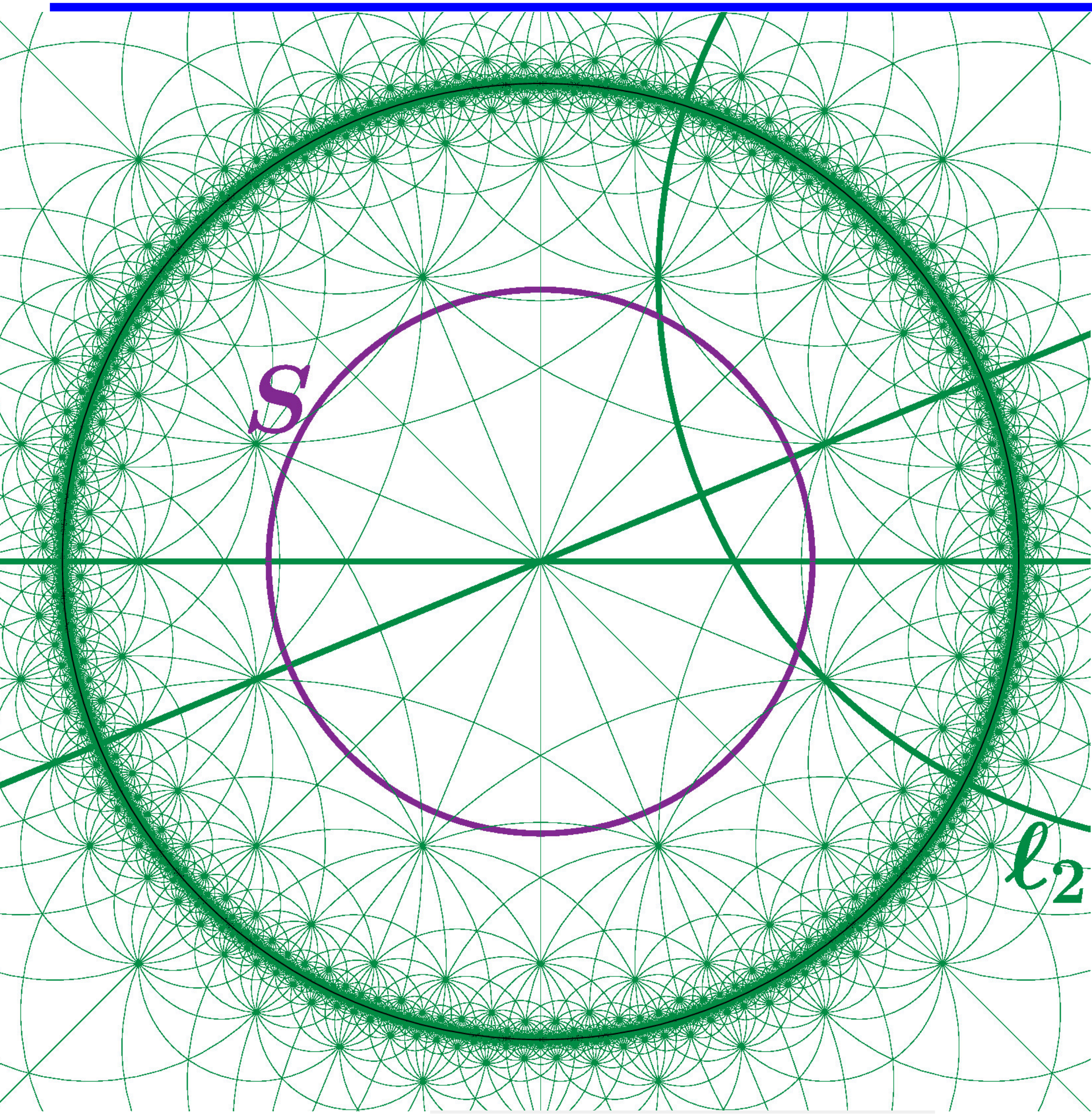

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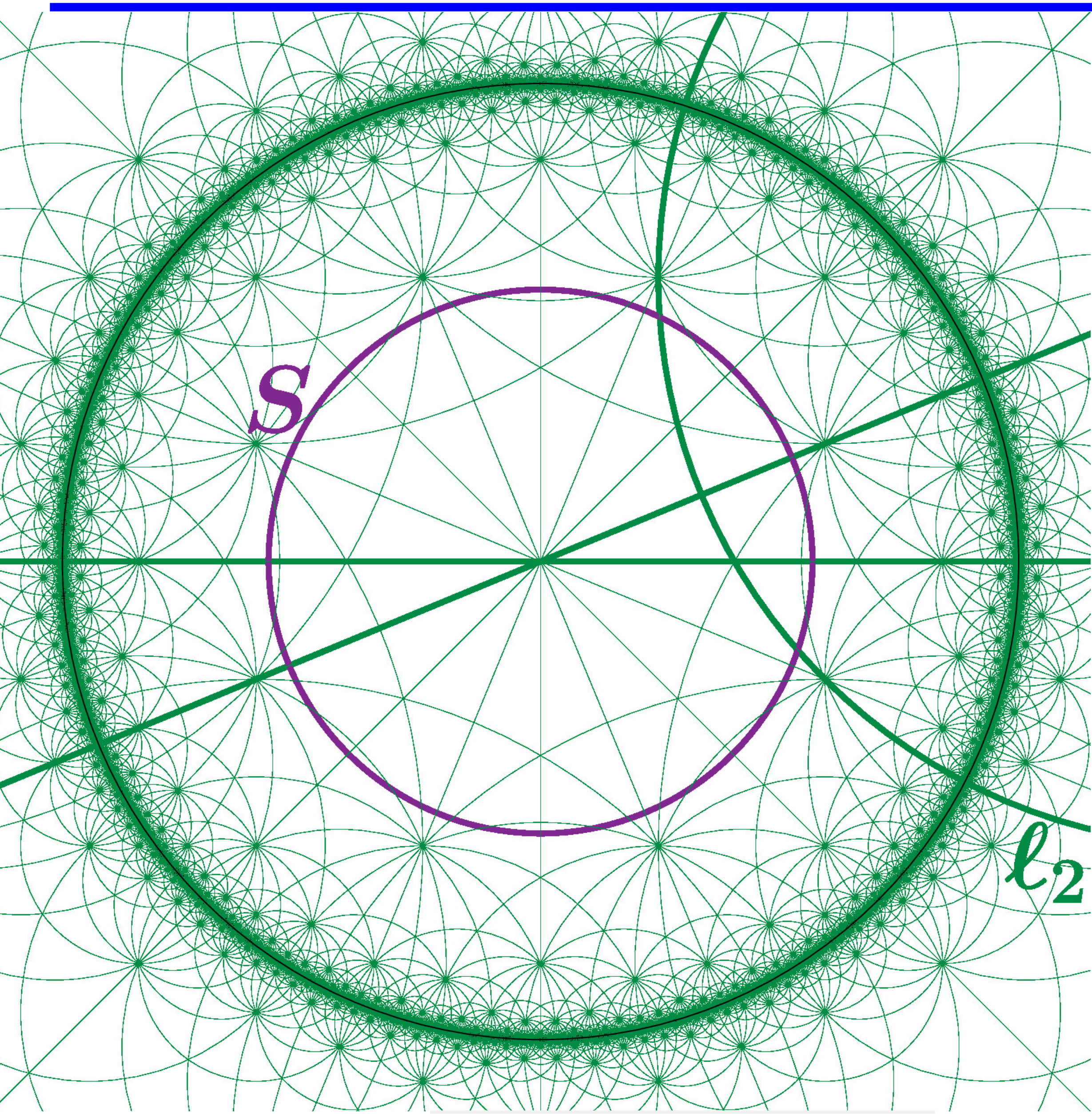

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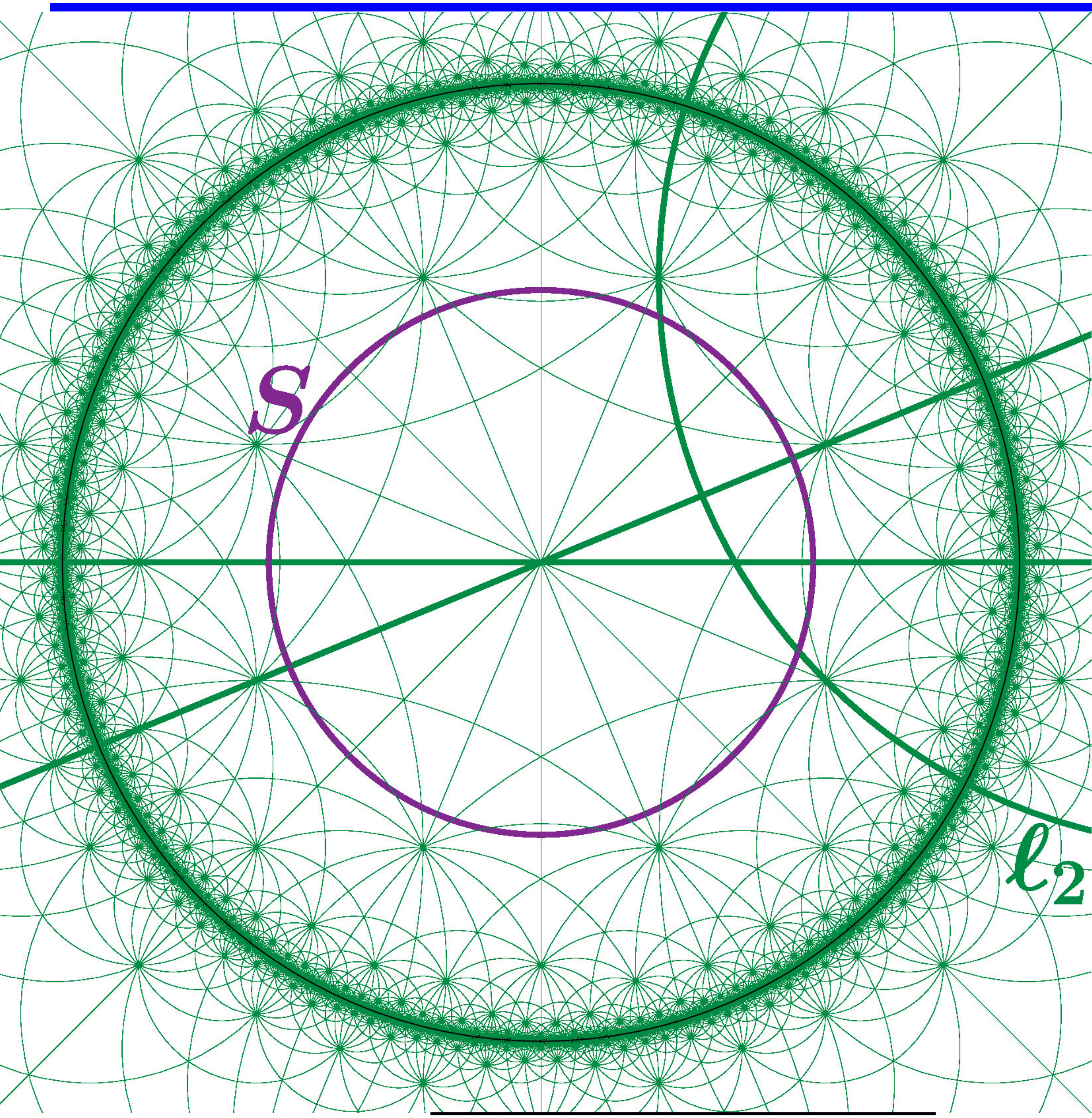

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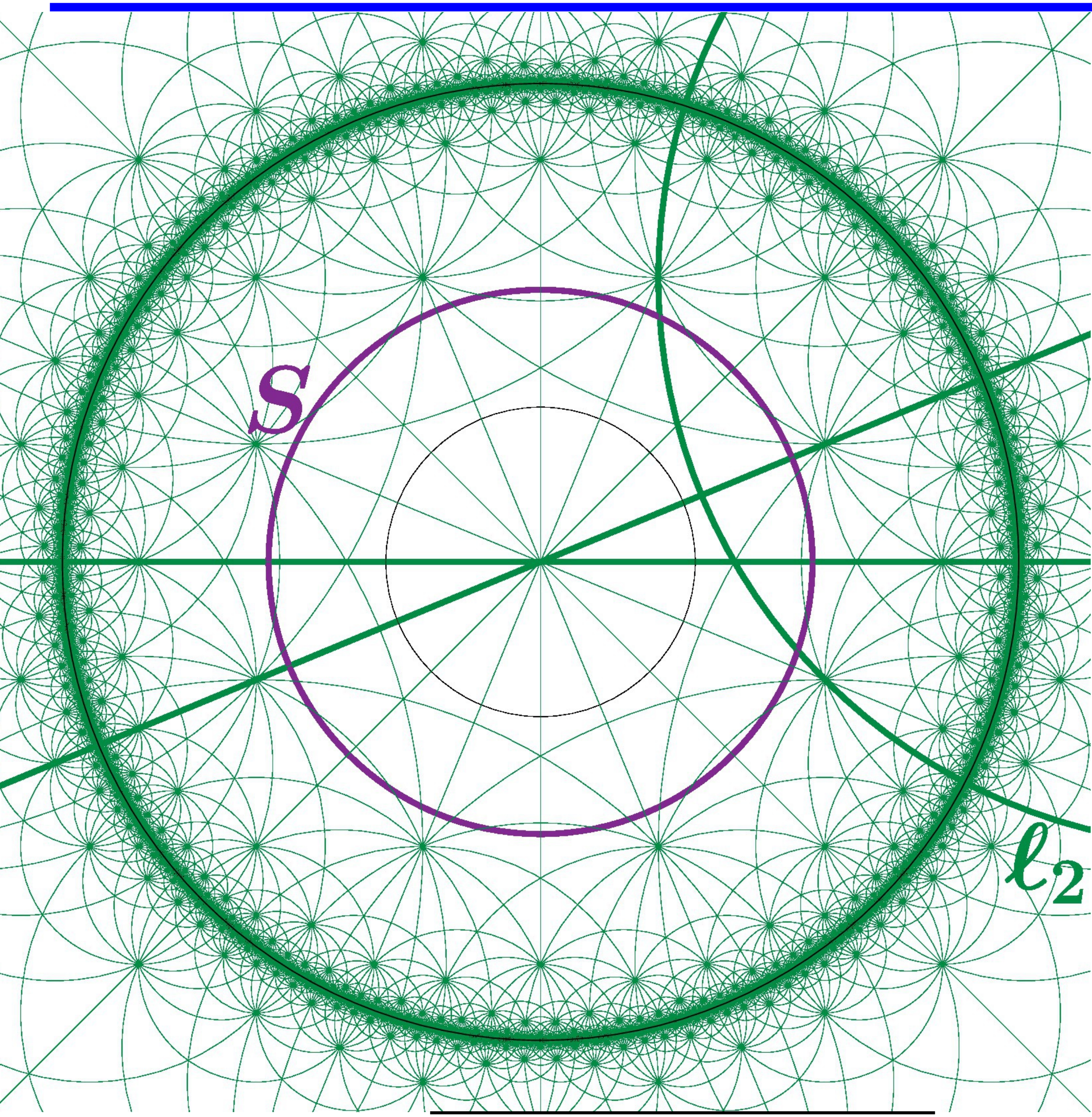

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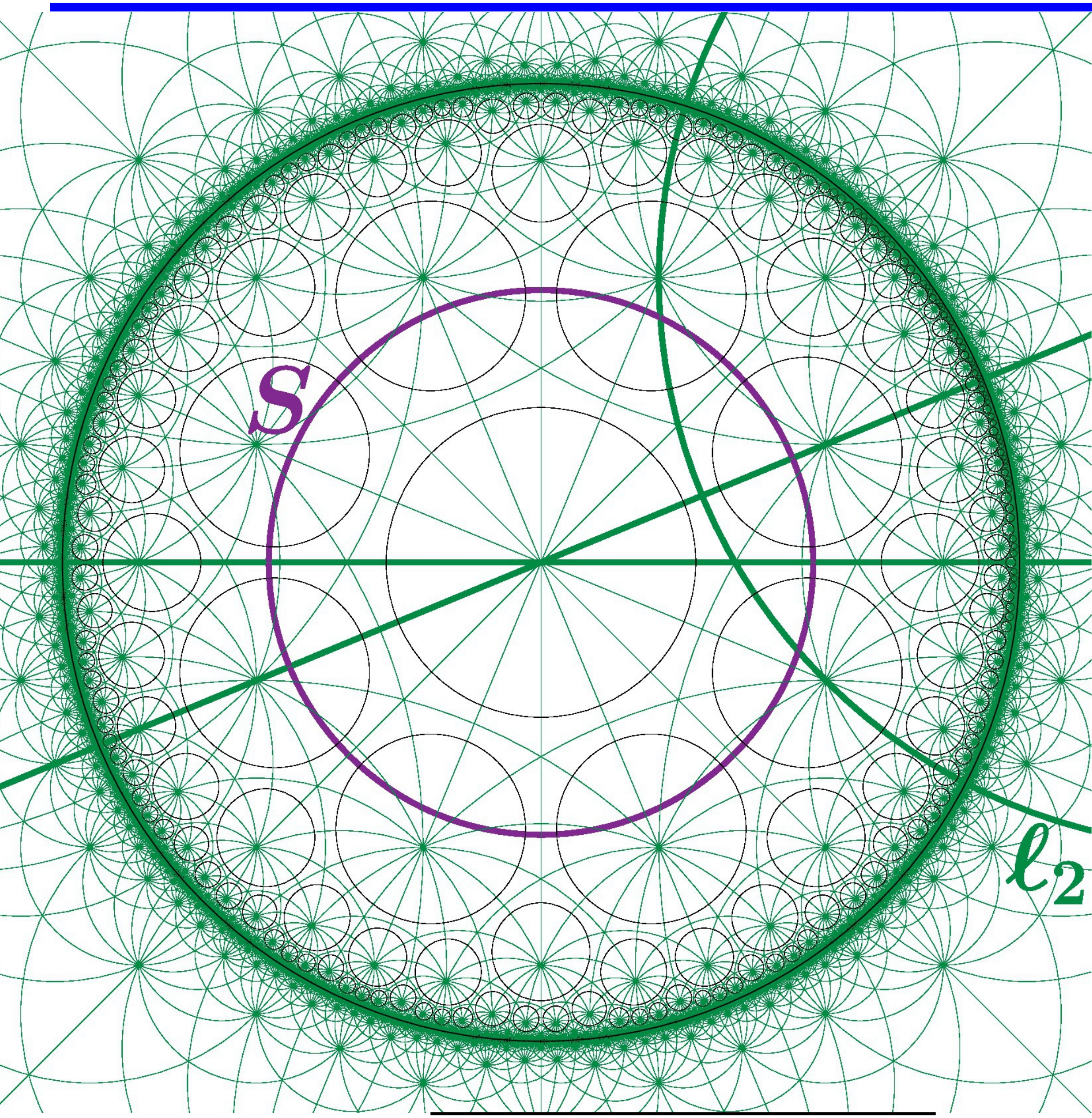

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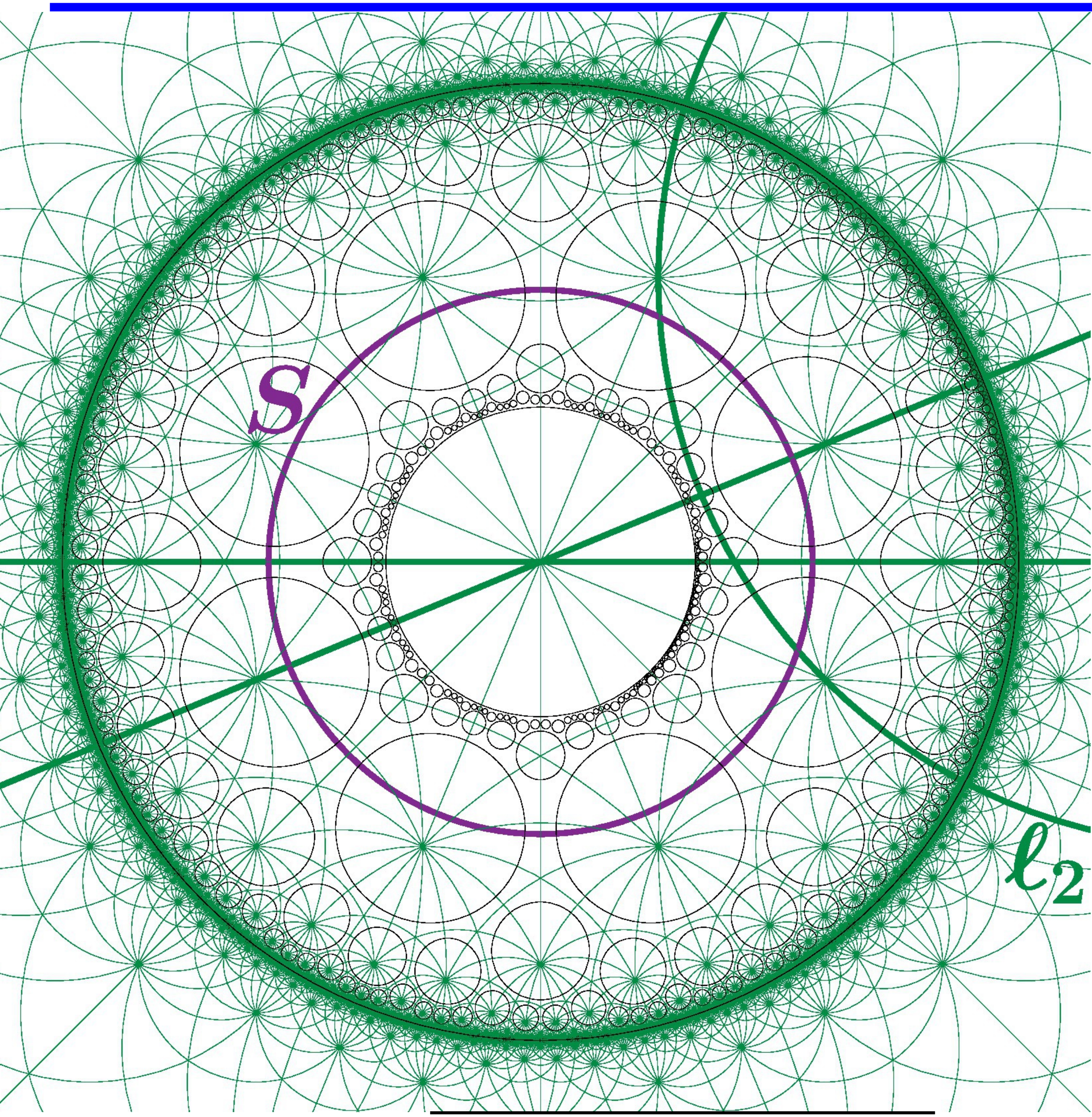

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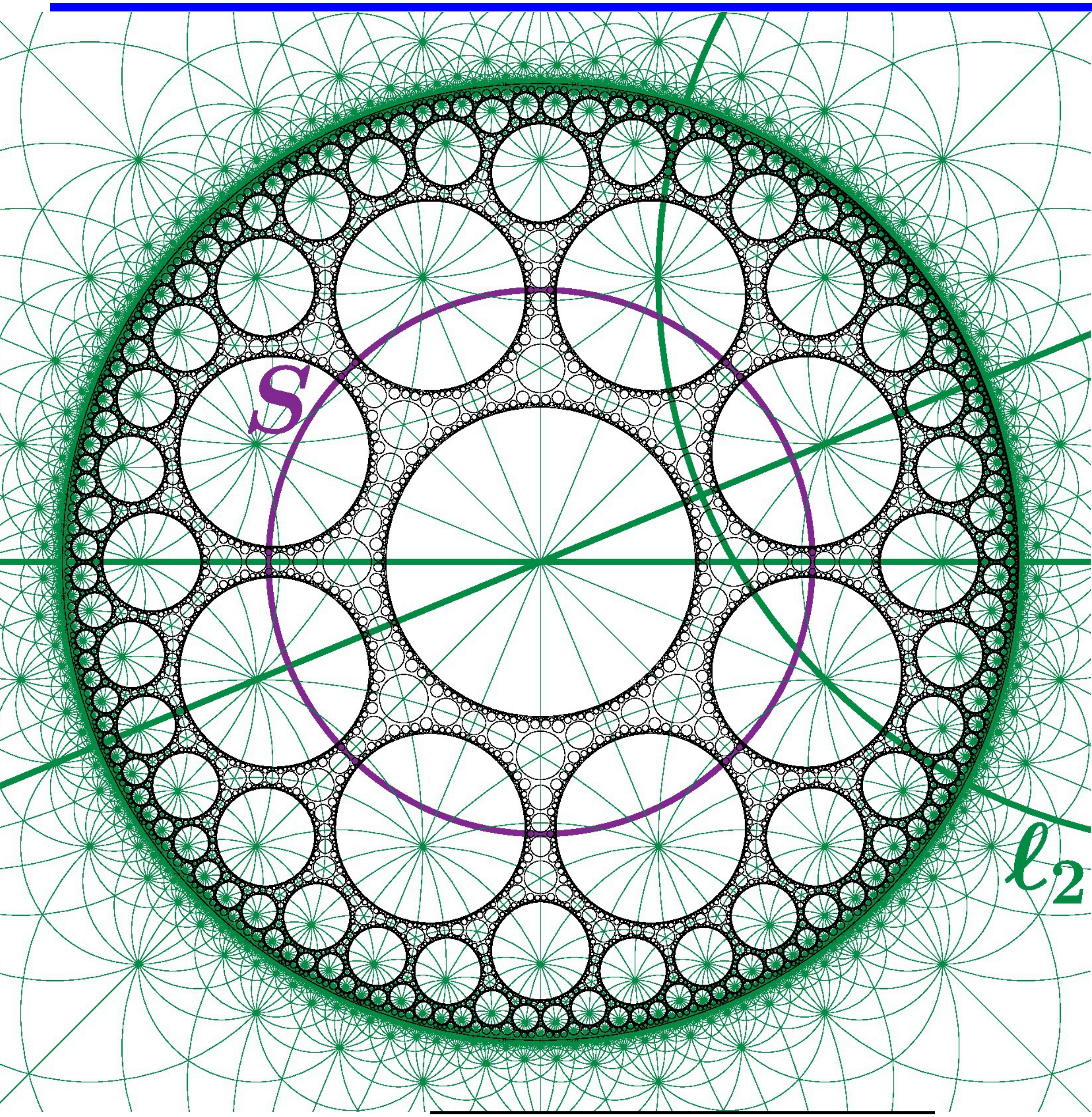

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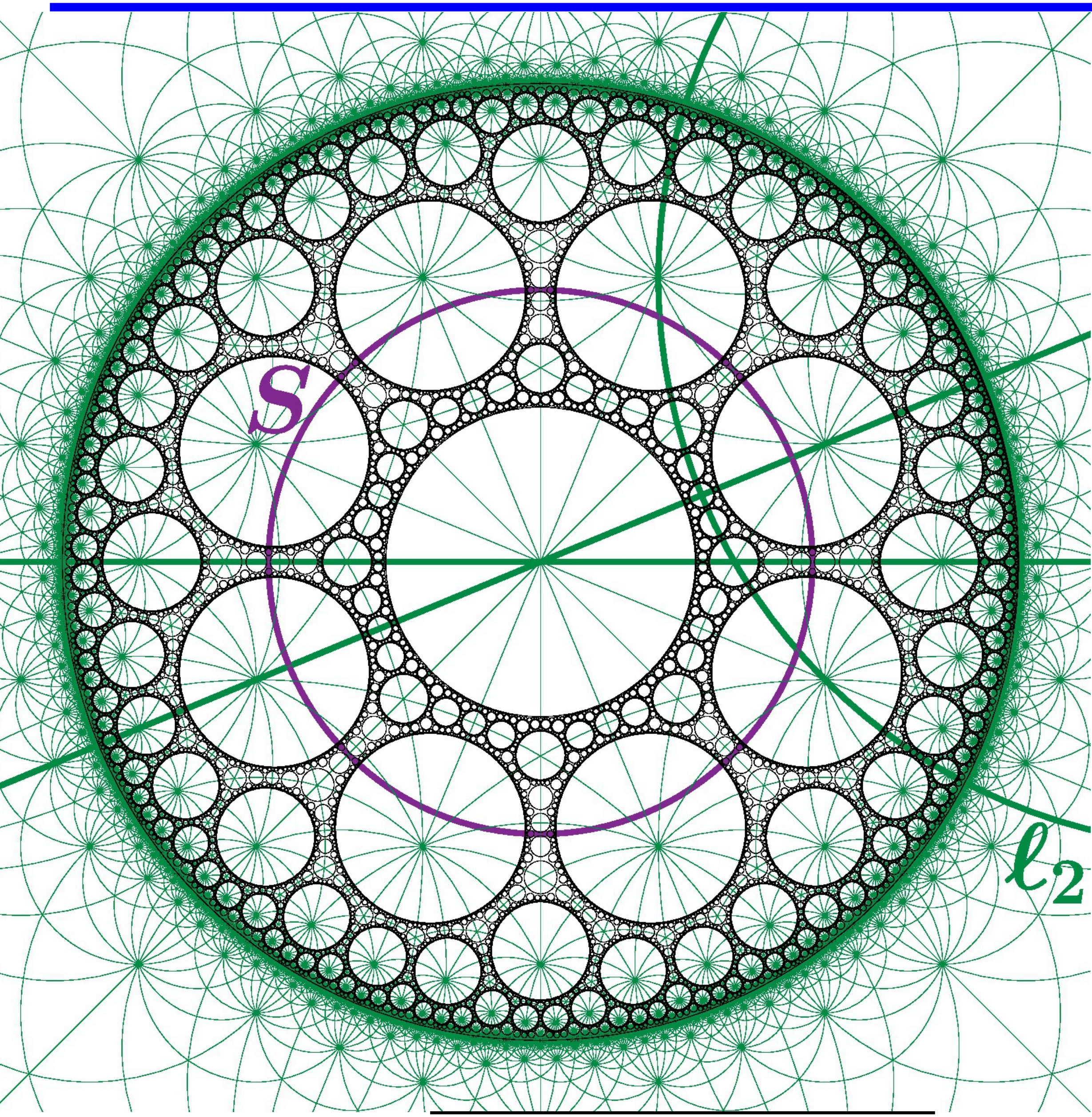

▷ $m > 6$ $\left(\frac{\pi}{2} + \frac{\pi}{3} + \frac{\pi}{m} < \pi\right)$

▷ $\{\ell_k\}_{k=1}^{3}$: $\mathbb{B}^2$-geodesics, form $\triangle$, angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{m}$

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$\rightsquigarrow \mathbb{B}^2 = \bigcup_{\tau \in \Gamma_m} \tau(\triangle_{\ell_1 \ell_2 \ell_3})$

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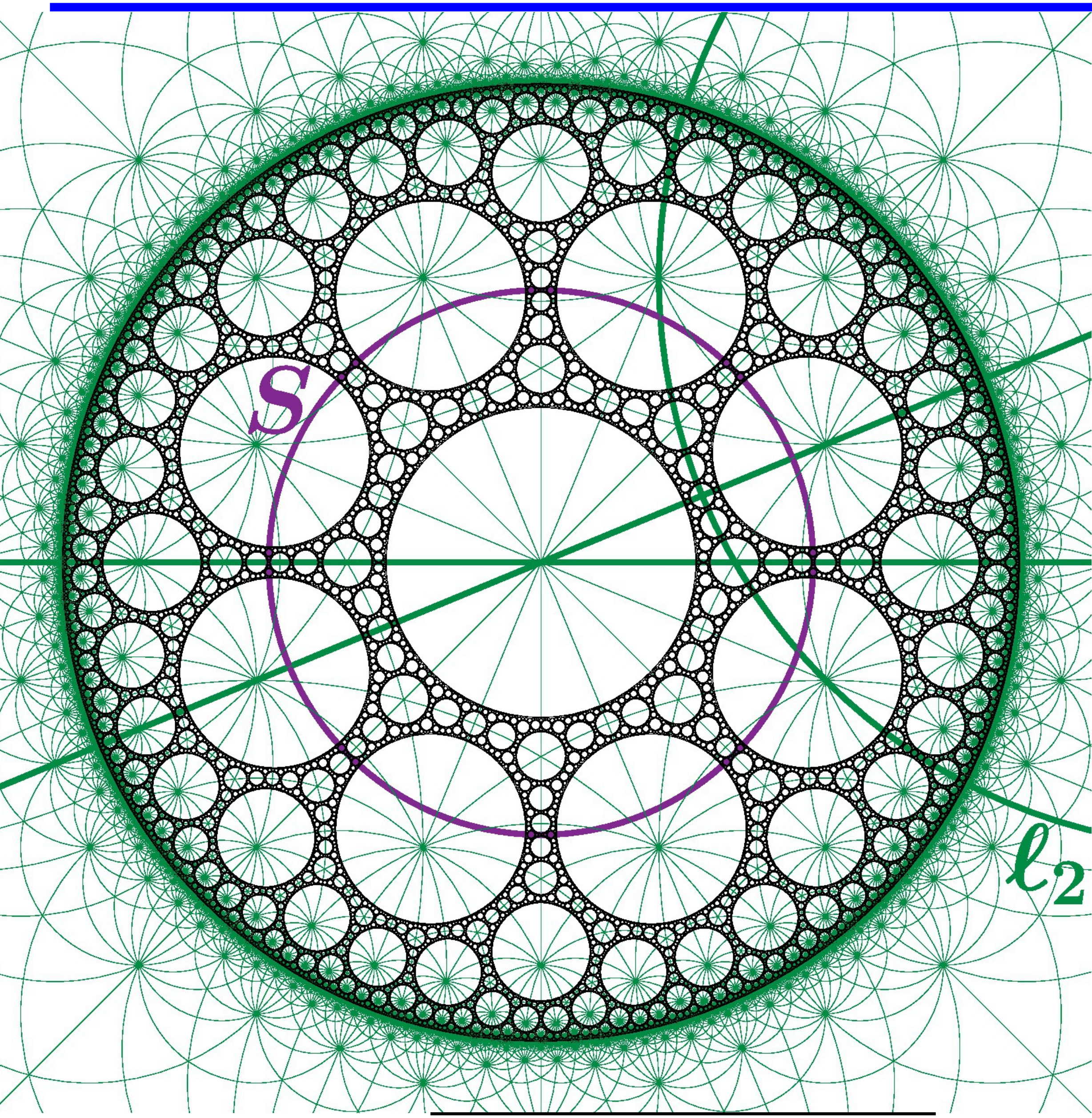

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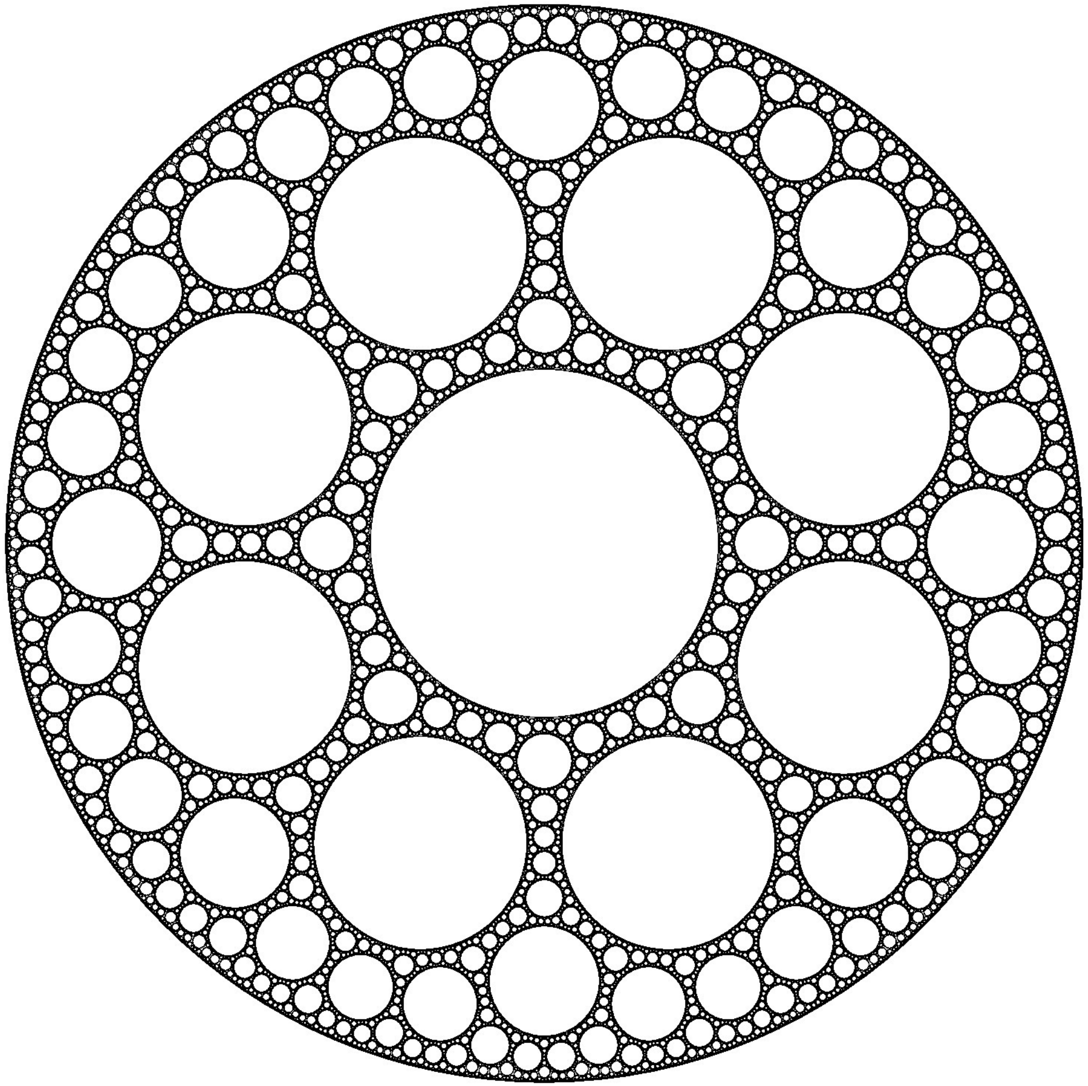

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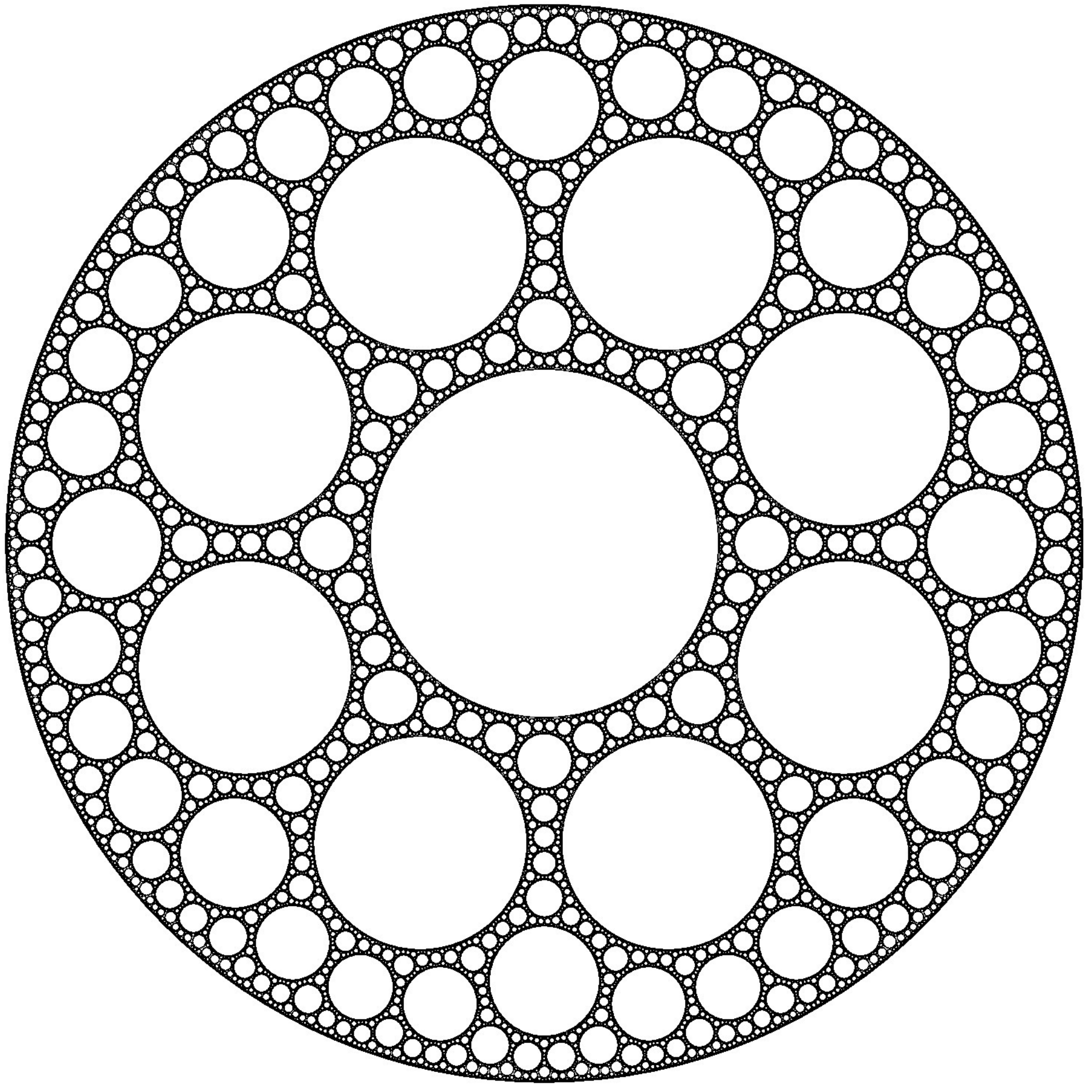

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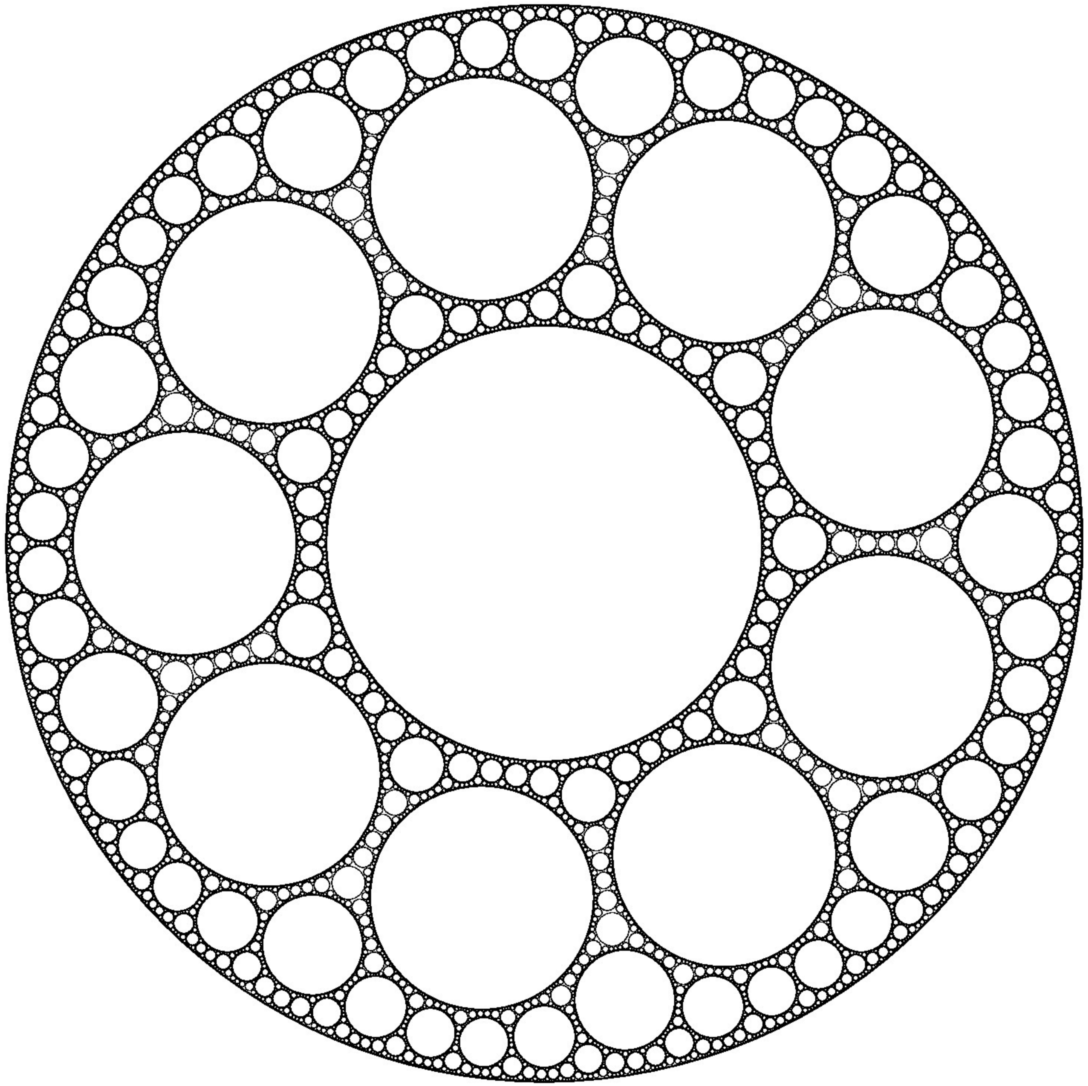

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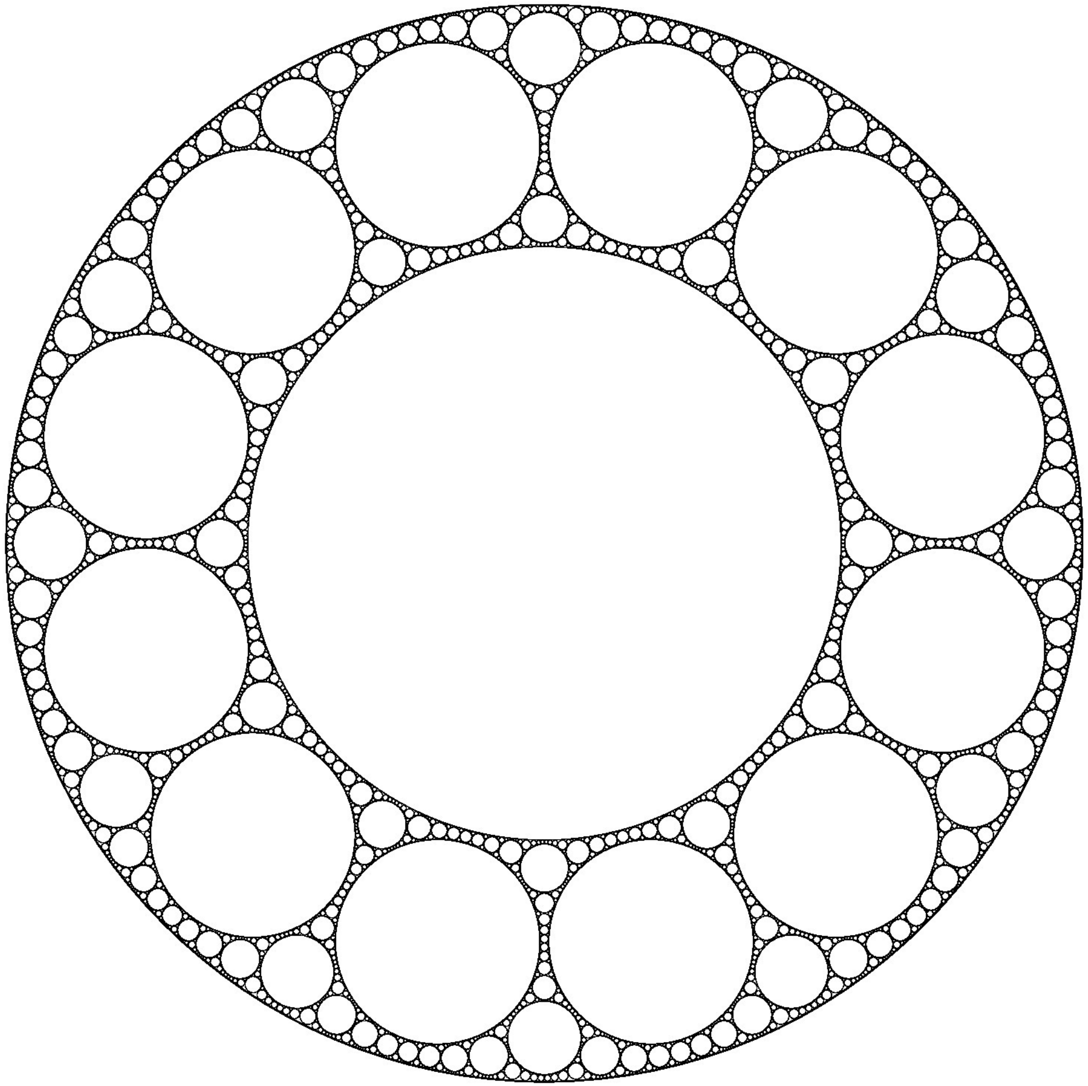

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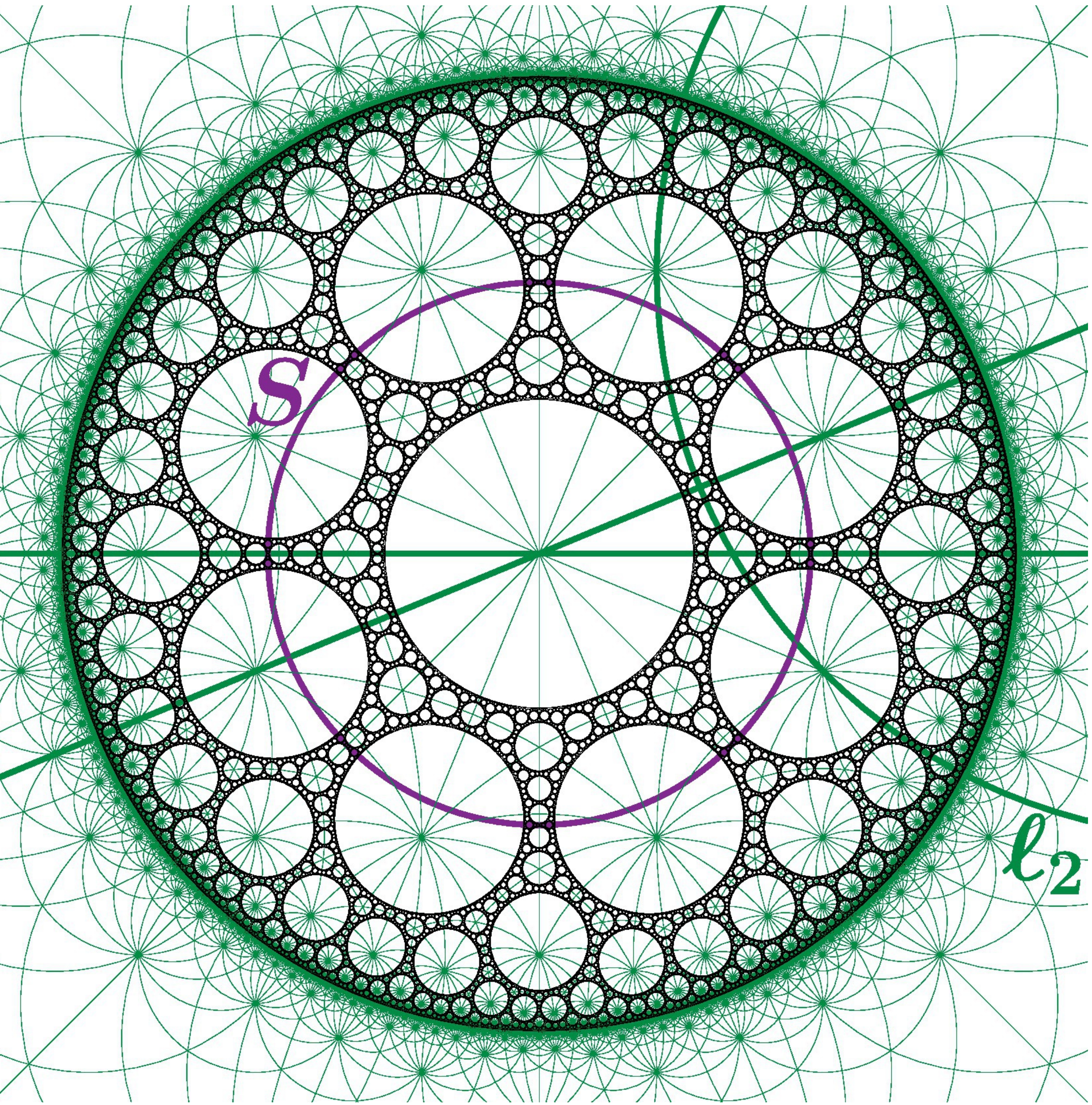

▷ $m > 6$ $\left(\frac{\pi}{2}+\frac{\pi}{3}+\frac{\pi}{m} < \pi\right)^{5/12}$

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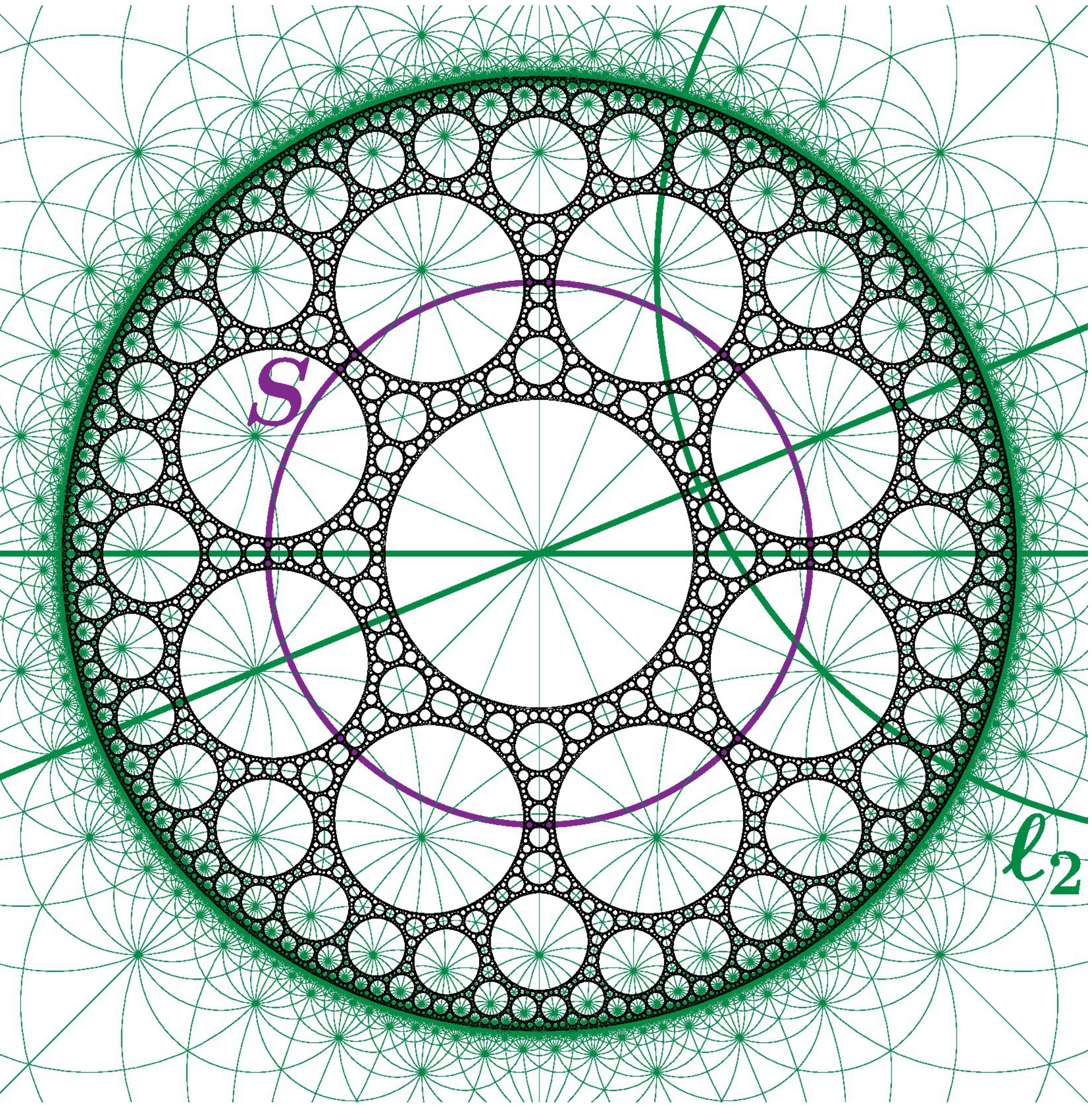

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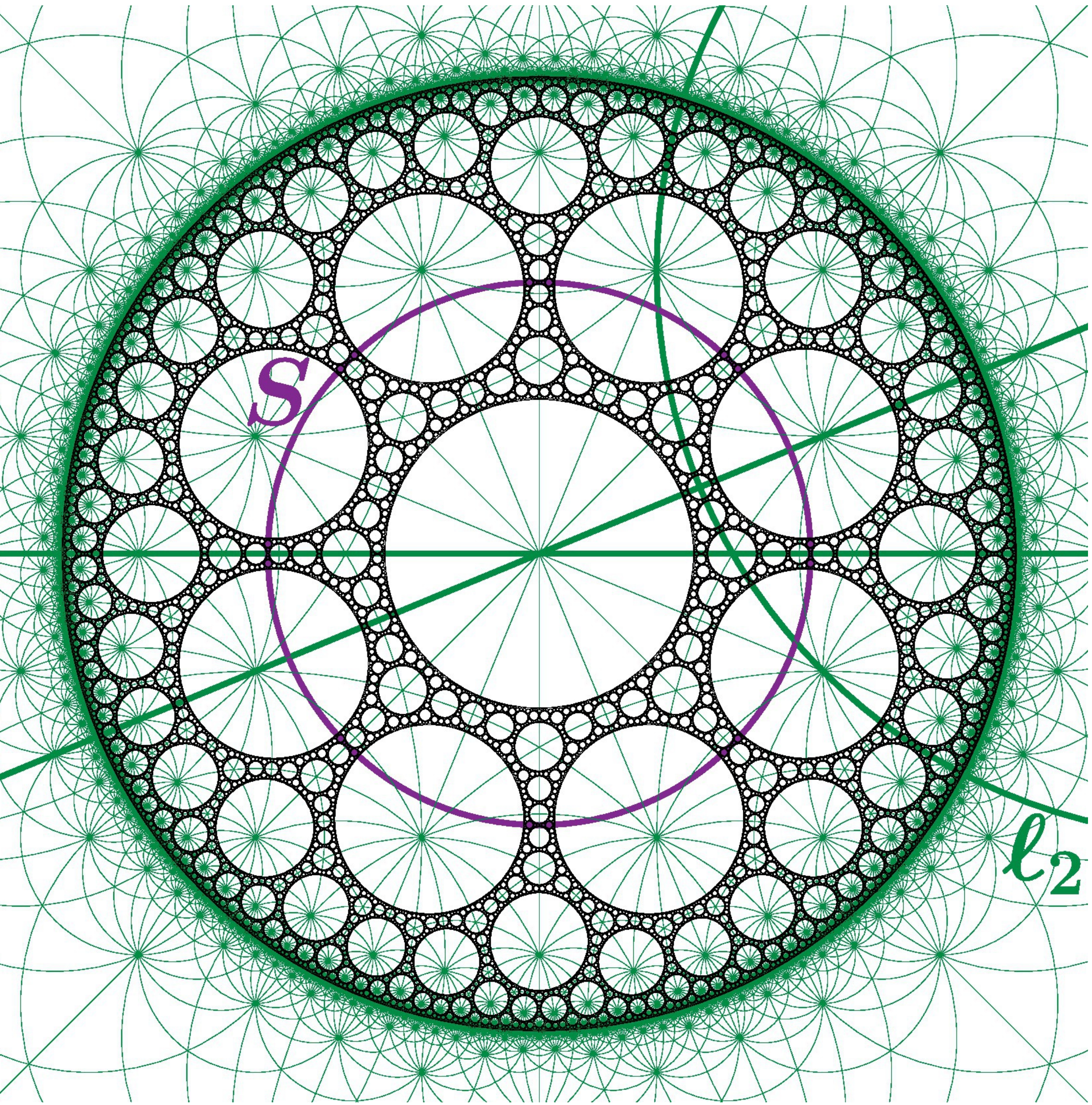

5/12

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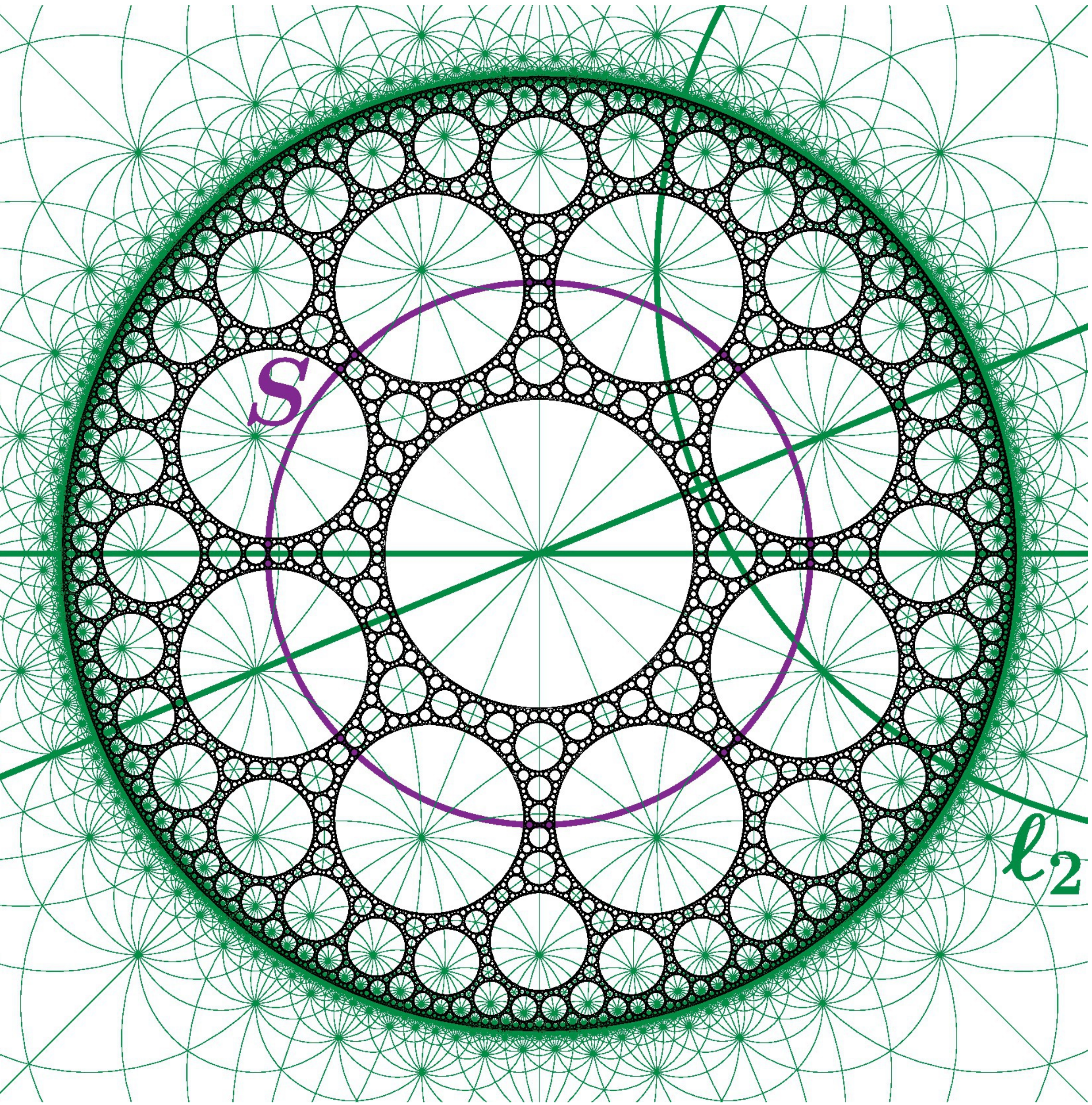

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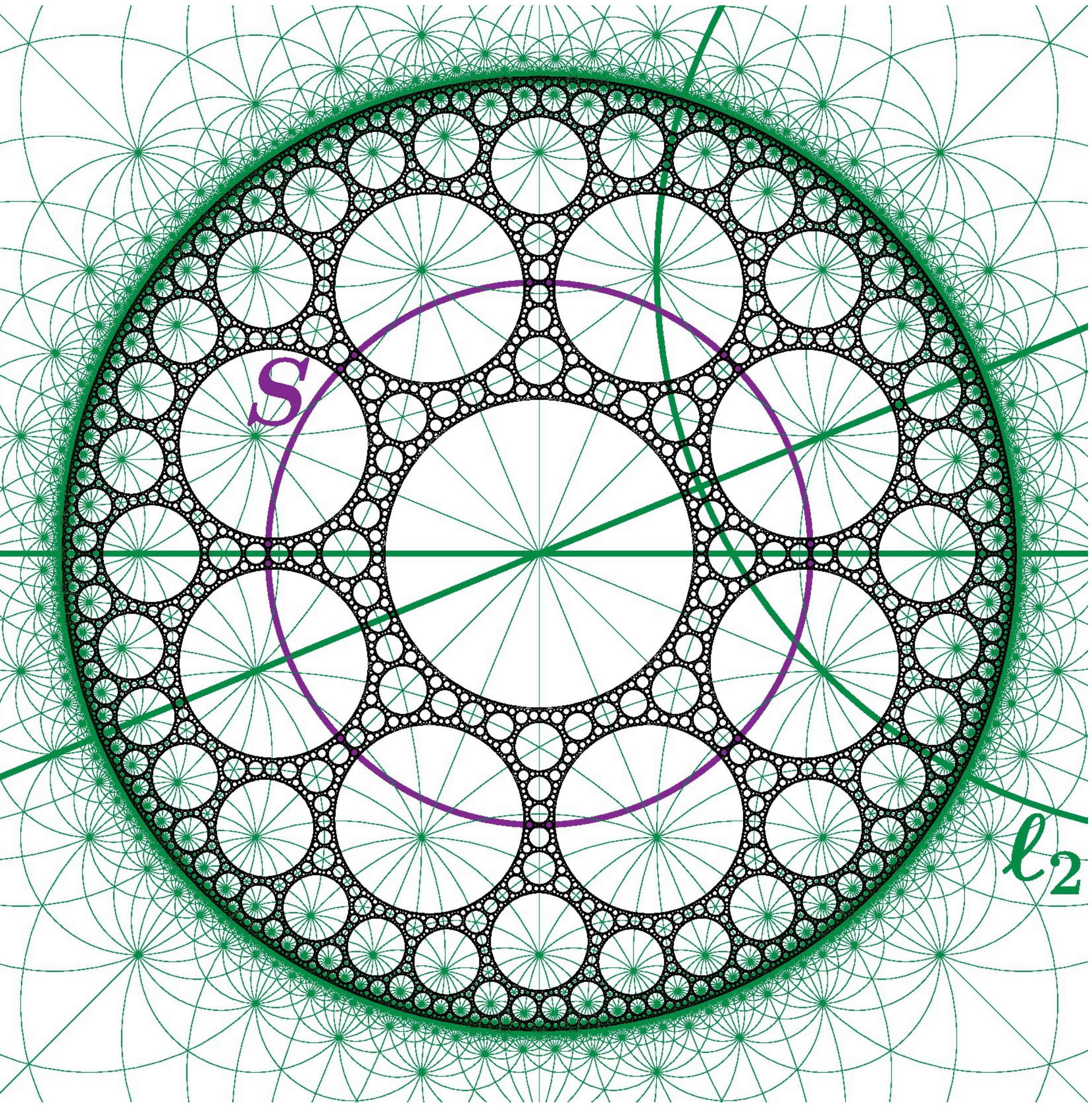

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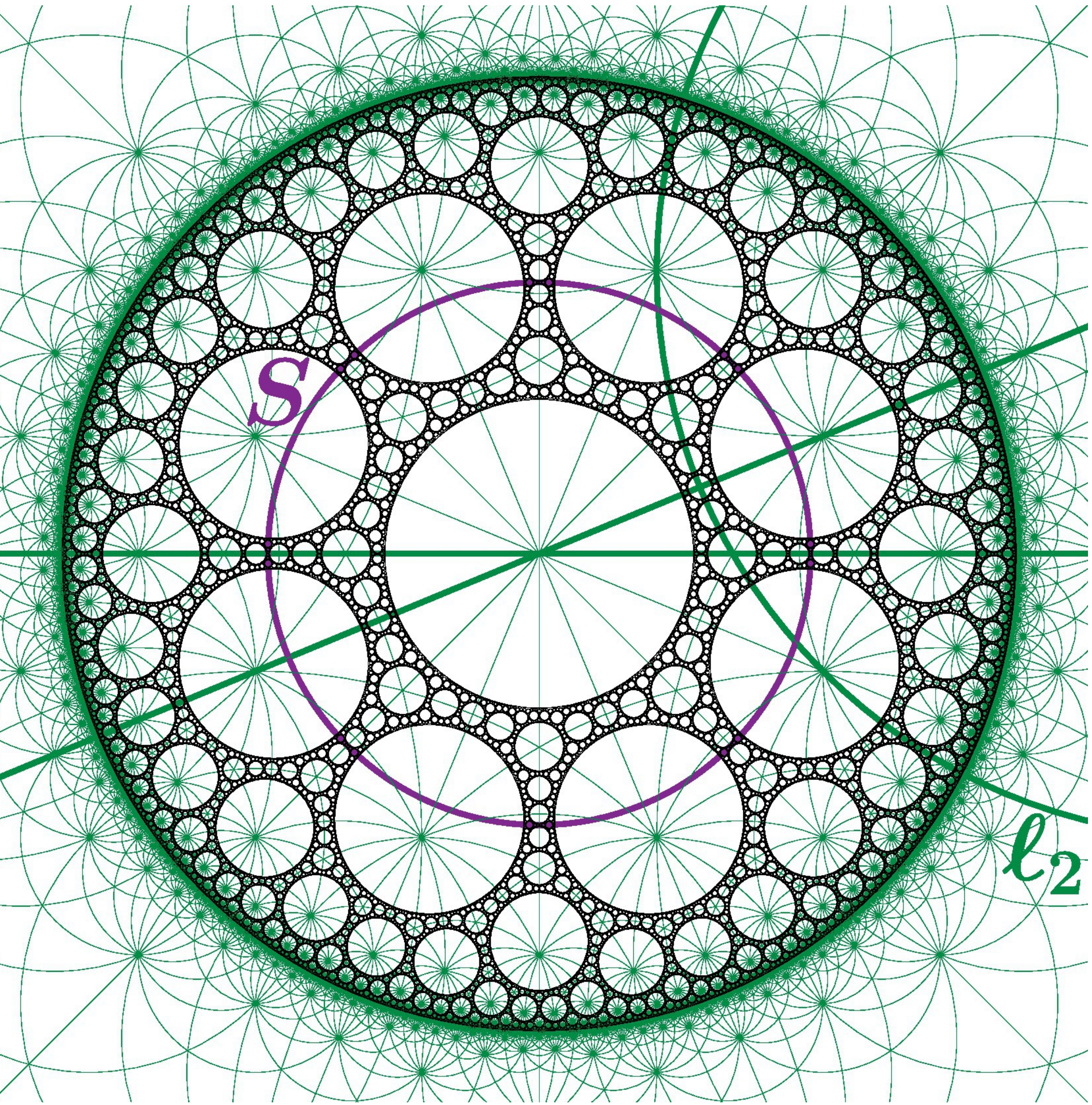
$\partial_\infty G$ is Ahlfors reg., so $\mathcal{H}^d|_{\partial_\infty G}$ is finite & full support.



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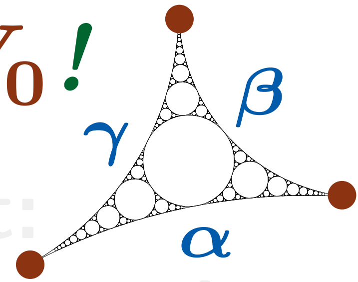
3 Results for Apollonian gasket: $K_{\alpha,\beta,\gamma} \overset{\text{harmonic}}{\underset{\text{embedding}}{\hookrightarrow}} \mathbb{C}$

Thm (K., *cf.* Teplyaev '04). $^{\exists 1}(\mathcal{E}^{\alpha,\beta,\gamma},\mathcal{F}^{\alpha,\beta,\gamma})$: str. local, irreducible, regular symmetric Dirichlet form over $K_{\alpha,\beta,\gamma}$, h_x, h_y are $\mathcal{E}^{\alpha,\beta,\gamma}$-*harmonic on* $K_{\alpha,\beta,\gamma} \setminus V_0$!



Rmk. Choice of a reference measure is irrelevant: $\mathcal{C}^{\alpha,\beta,\gamma} := \mathcal{F}^{\alpha,\beta,\gamma} \cap C(K_{\alpha,\beta,\gamma})$ and $\mathcal{E}^{\alpha,\beta,\gamma}|_{\mathcal{C}^{\alpha,\beta,\gamma}}$ are unique.

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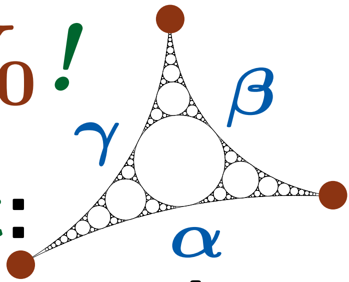

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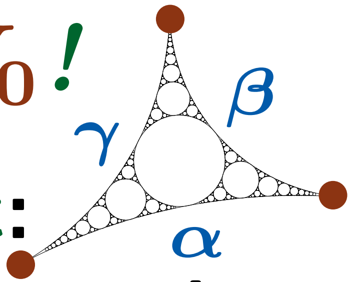

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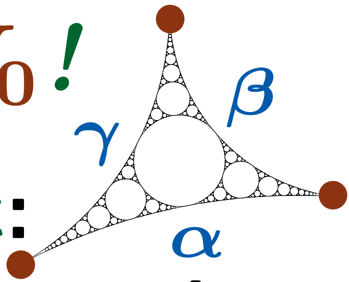

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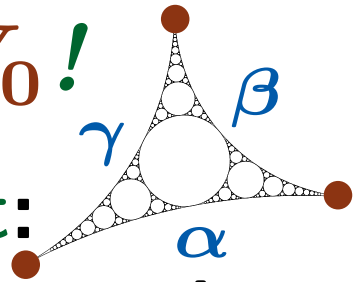
3 Results for Apollonian gasket: $K_{\alpha,\beta,\gamma} \overset{\text{harmonic}}{\underset{\text{embedding}}{\hookrightarrow}} \mathbb{C}$

Thm (K., *cf.* Teplyaev '04). $^{\exists 1}(\mathcal{E}^{\alpha,\beta,\gamma}, \mathcal{F}^{\alpha,\beta,\gamma})$: str. local, irreducible, regular symmetric Dirichlet form over $K_{\alpha,\beta,\gamma}$, h_x, h_y are $\mathcal{E}^{\alpha,\beta,\gamma}$-*harmonic on* $K_{\alpha,\beta,\gamma} \setminus V_0$!



Rmk. Choice of a reference measure is irrelevant: $\mathcal{C}^{\alpha,\beta,\gamma} := \mathcal{F}^{\alpha,\beta,\gamma} \cap C(K_{\alpha,\beta,\gamma})$ and $\mathcal{E}^{\alpha,\beta,\gamma}|_{\mathcal{C}^{\alpha,\beta,\gamma}}$ are unique.

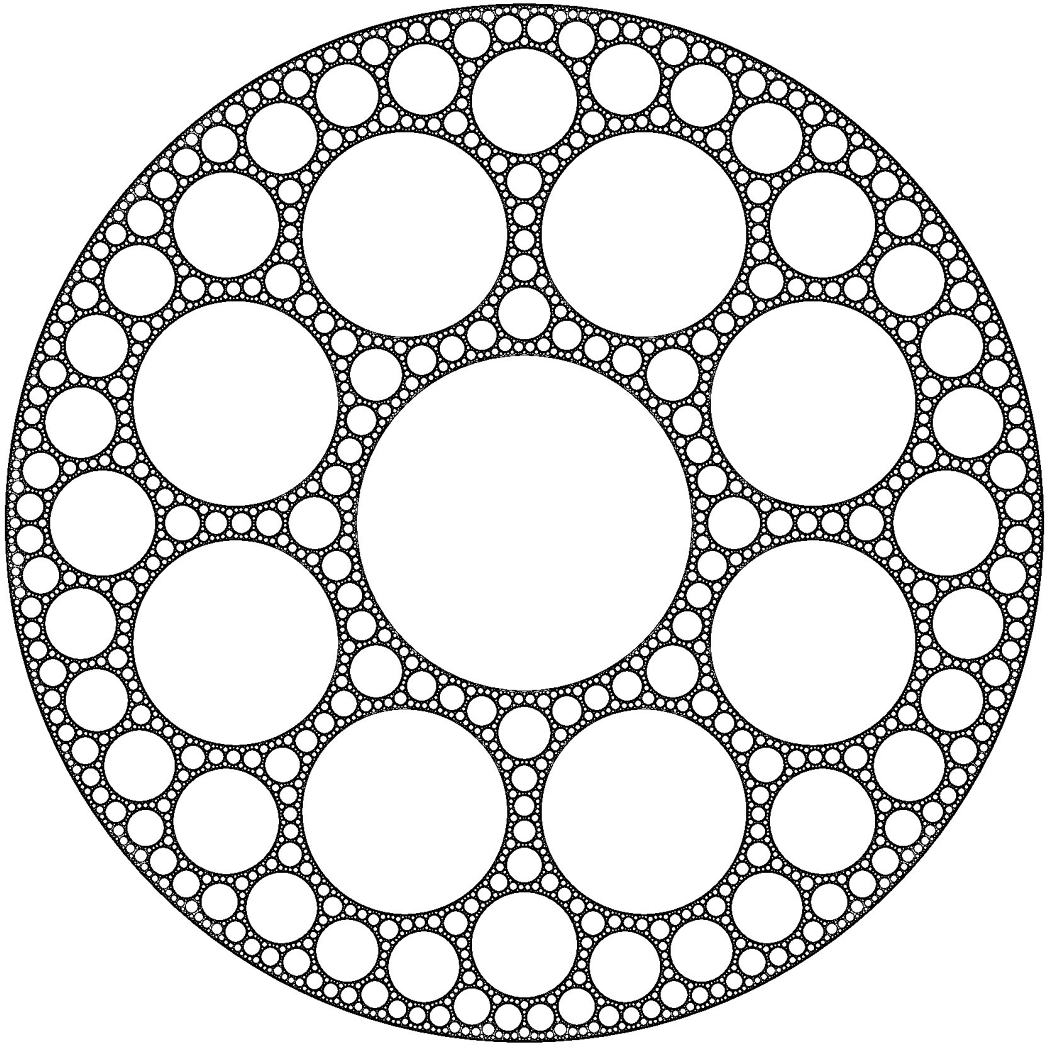
Thm (K.). $\mathrm{LIP}|_{K_{\alpha,\beta,\gamma}}$ is a core of $(\mathcal{E}^{\alpha,\beta,\gamma}, \mathcal{F}^{\alpha,\beta,\gamma})$, and $^{\forall} u \in \mathrm{LIP}$, $\mathcal{E}^{\alpha,\beta,\gamma}(u,u) = \sum_{C \subset_{\mathrm{arc}} K_{\alpha,\beta,\gamma}} \mathrm{rad}(C) \int_C |\nabla_C u|^2 d\mathrm{vol}_C$.

$\triangleright$ $\mu^{\alpha,\beta,\gamma} := \sum_{C \subset_{\mathrm{arc}} K_{\alpha,\beta,\gamma}} \mathrm{rad}(C) d\mathrm{vol}_C$: volume meas. (NOT doubling!)

Thm (K.). $^{\exists} c_0 \in (0,\infty)$, $^{\forall} \alpha, \beta, \gamma \in (0,\infty)$,

$$\lim_{\lambda \to \infty} \#\{n \in \mathbb{N} \mid \lambda_n^{\alpha,\beta,\gamma} \le \lambda\} / \lambda^{d/2} = c_0 \mathcal{H}^d(K_{\alpha,\beta,\gamma}).$$

4 Laplacian on the limit set $\partial_\infty G$ of $G = G_m$



cf. $\mathcal{E}^{\alpha,\beta,\gamma}(u,u) = \sum_{C \subset_{\mathrm{arc}} K_{\alpha,\beta,\gamma}} \mathrm{rad}(C) \int_C |\nabla_C u|^2 \, d\mathrm{vol}_C,$

$$\mu^{\alpha,\beta,\gamma} = \sum_{C \subset_{\mathrm{arc}} K_{\alpha,\beta,\gamma}} \mathrm{rad}(C) \, d\mathrm{vol}_C.$$

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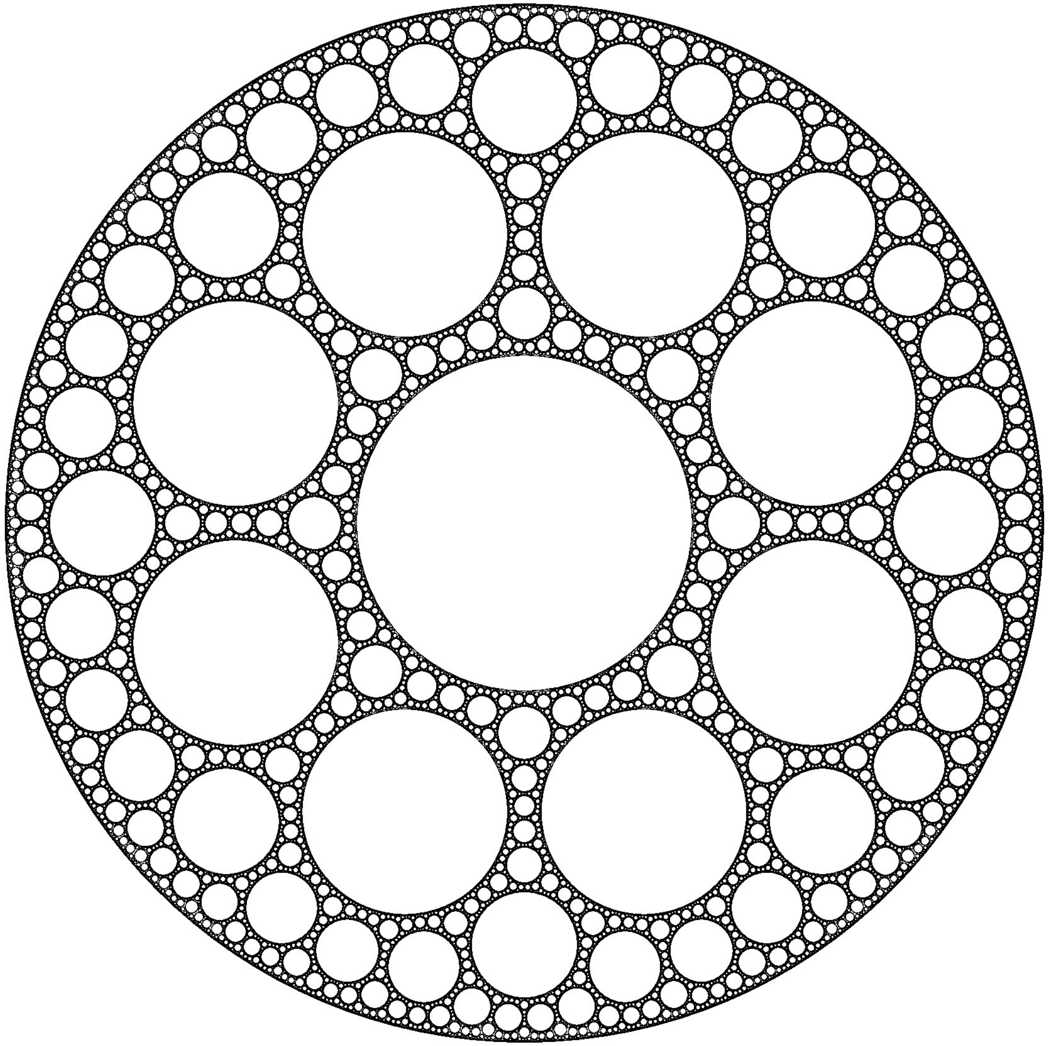
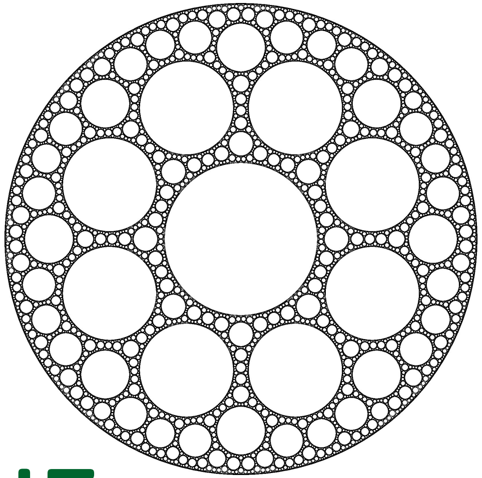


▷ $\mathcal{G} := \{g \in \textbf{Möb}(\widehat{\mathbb{C}}) \mid g^{-1}(\infty) \in \widehat{\mathbb{C}} \setminus \overline{\mathbb{B}^2}\}$

▷ $K_0 := \mathbb{B}^2 \cap \partial_\infty G, \quad K_g := g(K_0)$ $\left(\begin{array}{l} g \in \mathcal{G} \text{ represents} \\ \text{choice of initial } \triangle \end{array} \right)$

$\triangleright\ \mathcal{G} := \{g \in \textsf{Möb}(\widehat{\mathbb{C}}) \mid g^{-1}(\infty) \in \widehat{\mathbb{C}} \setminus \overline{\mathbb{B}^2}\}$



$\triangleright\ K_0 := \mathbb{B}^2 \cap \partial_\infty G, \quad K_g := g(K_0)$

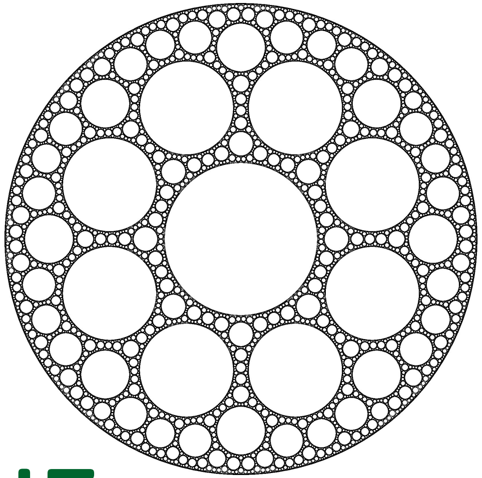
For each $g \in \mathcal{G}$, on $K_g = g(K_0)$ DEFINE:

$\triangleright\ \nu^g := \sum_{C \subset_{\mathrm{arc}} K_g} \mathrm{rad}(C)\, d\mathrm{vol}_C$ **(NOT doubling!)**

$\triangleright\ {}^\forall u \in \mathrm{LIP}|_{K_g},\ \mathcal{E}^g(u,u) := \sum_{C \subset_{\mathrm{arc}} K_g} \mathrm{rad}(C) \int_C |\nabla_C u|^2\, d\mathrm{vol}_C$ (*cf.* Osada '07)

Prop. On $L^2(K_g, \nu^g)$, $(\mathcal{E}^g, \mathrm{LIP}_c(K_g))$ is closable & its closure $(\mathcal{E}^g, \mathcal{F}_g)$ is a strongly local regular Dirichlet form.

Prop. The inclusion map $\iota : K_g \hookrightarrow \mathbb{R}^2$ is $\mathcal{E}^g$-*harmonic*.
(uniqueness NOT known)



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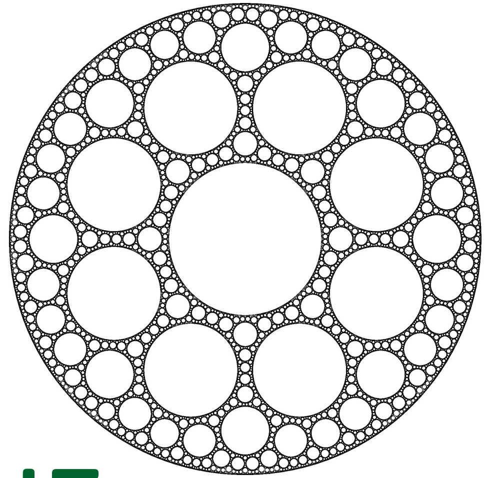

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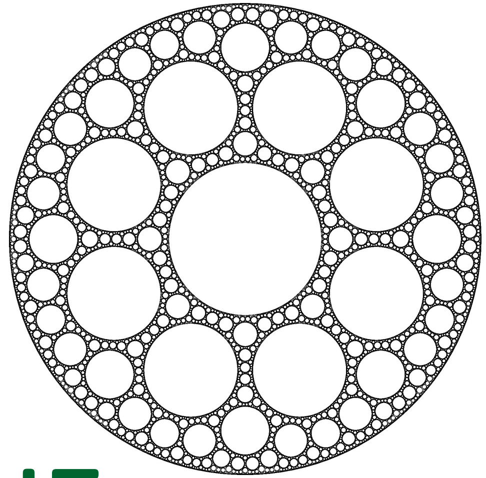

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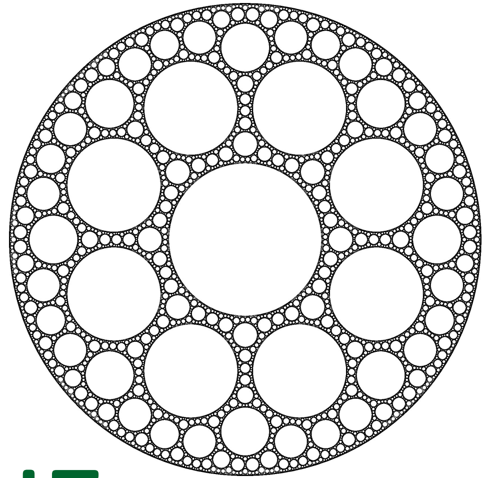

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▷ $\{\lambda_n^{g,U}\}_{n \in \mathbb{N}}$: the eigenvalues of $-\Delta_{(U, \nu^g, \mathcal{E}^g, \mathcal{F}_g)}$

▷ $\mathcal{N}_{g,U}(\lambda) := \#\{n \in \mathbb{N} \mid \lambda_n^{g,U} \leq \lambda\}$ ($\emptyset \neq U \subset K_g$ open)

Thm (K.). $^\exists c_m \in (0, \infty)$, $^\forall g \in \mathcal{G}$, $_{\emptyset \neq}^{\ \ \forall} U \in K_g$ open,

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Rmk. $\sum_n e^{-t\lambda_n^{g,U}} = \int_U p_t^{g,U}(x,x)\,d\nu^g(x) \overset{t\downarrow 0}{\sim} c\mathcal{H}^d(U)t^{-d/2}$

$\Leftrightarrow$ **Thm**, BUT

$\triangleright$ $g \in \mathcal{G}$ (represents choice of the initial $\mathbb{B}^2$-$\triangle$)

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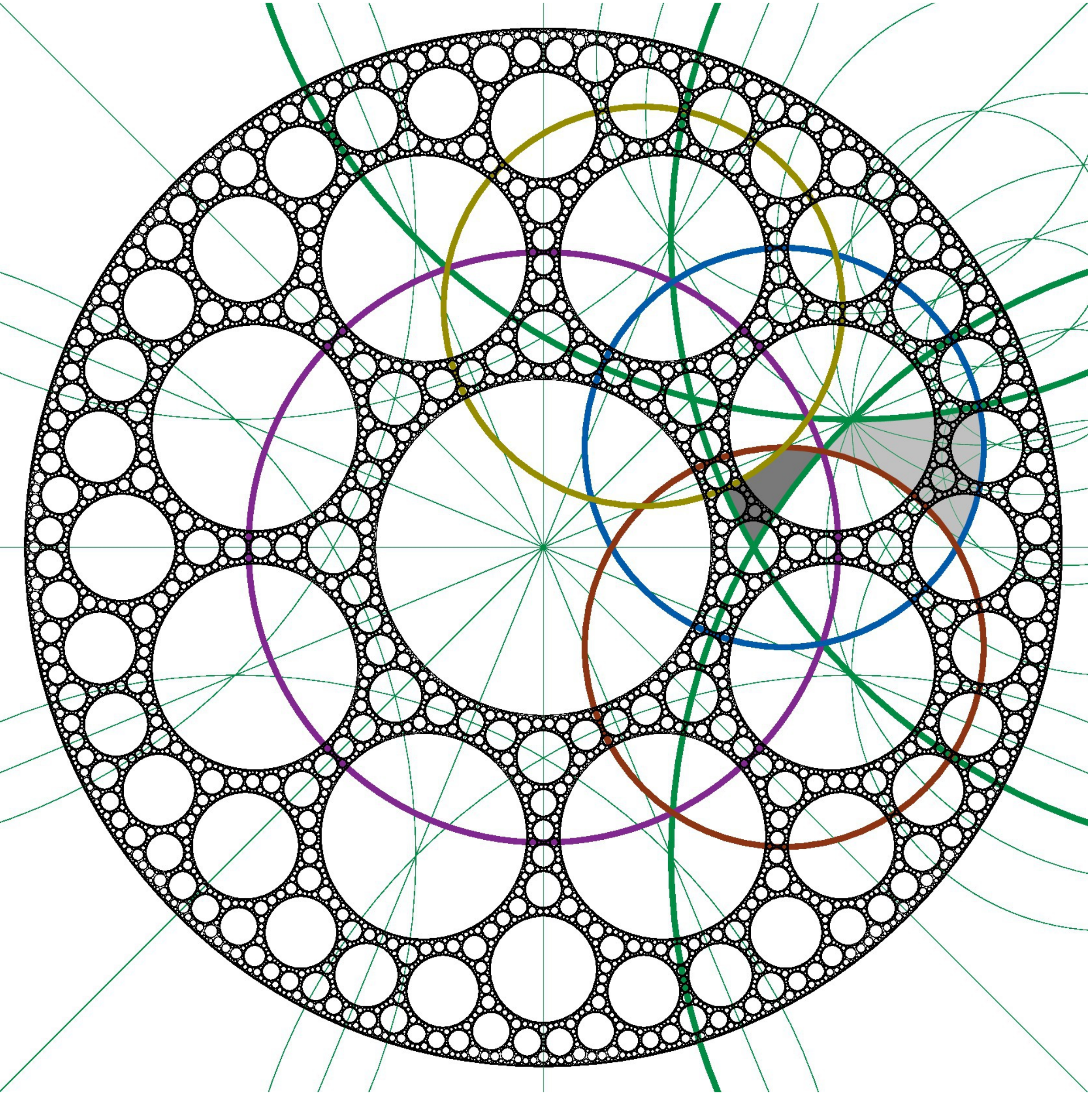
$\Leftrightarrow$ **Thm**, BUT $p_t^{g,U}(x,x) \asymp_{c_x} t^{-1/2}$ for ν^g-a.e. $x \in U$!

5 Ingredients of the proof of Weyl's asymptotics



- A "self-similar" decomp. ("fundamental domain" for the action $G \curvearrowright \partial_\infty G$)

$\triangleright \{\ell_k\}_{k=1}^3$: $\mathbb{B}^2$-geodesics, form $\triangle$, angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{m}$

$\triangleright\ \Gamma_m := \langle \{\mathrm{Inv}_{\ell_k}\}_{k=1}^3 \rangle$

$\rightsquigarrow \mathbb{B}^2 = \bigcup_{\tau \in \Gamma_m} \tau(\triangle_{\ell_1 \ell_2 \ell_3})$

- $S = S_m := \partial B_{\mathbb{B}^2}(0, {}^{\exists 1} r_m)$: $\mathrm{angle}(S, \ell_2) = \frac{\pi}{3}$.

$\triangleright\ G = G_m := \langle \Gamma_m, \mathrm{Inv}_S \rangle$, $\partial_\infty G_m := \overline{\bigcup_{g \in G_m} g(\partial \mathbb{B}^2)}$

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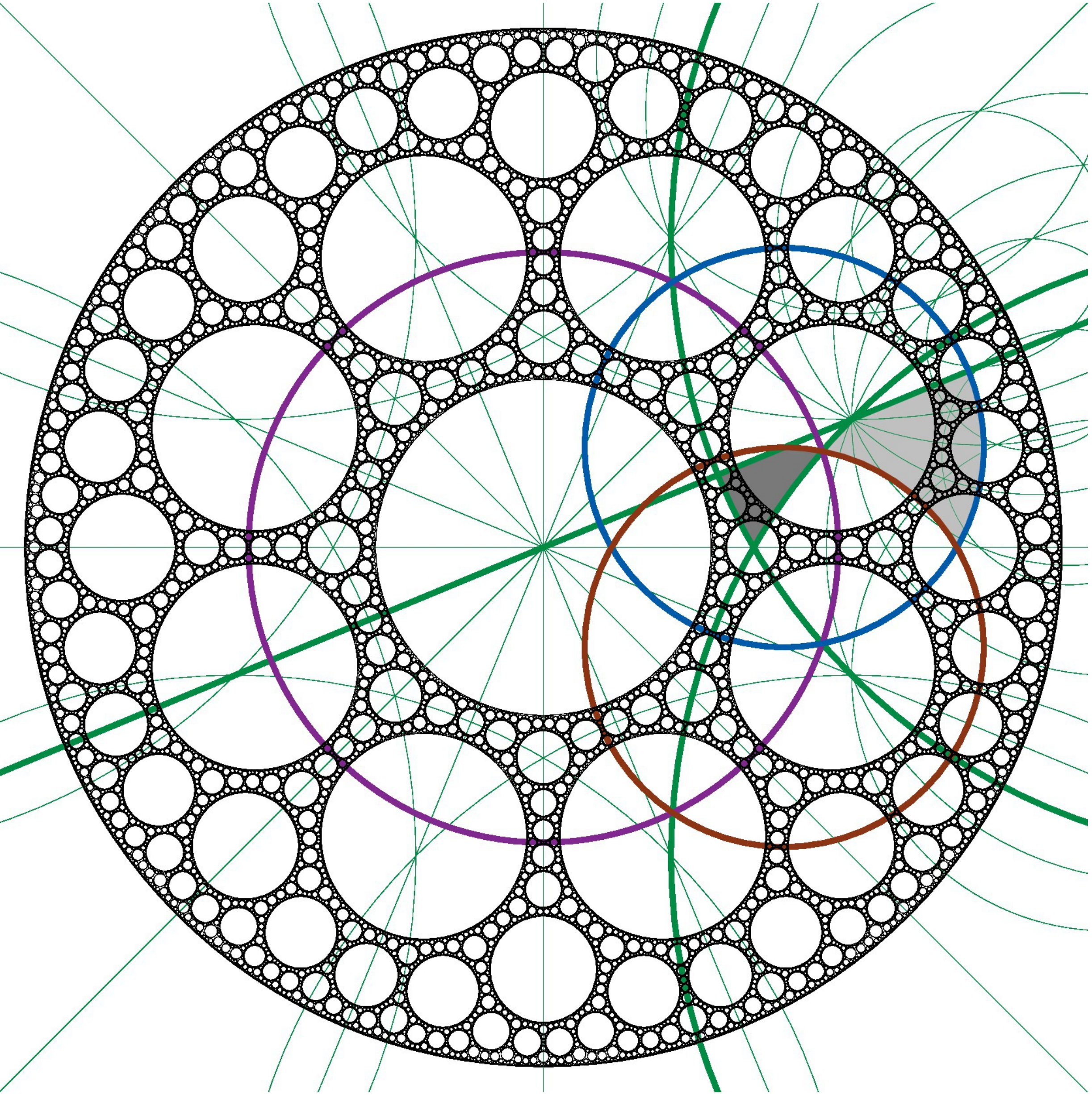

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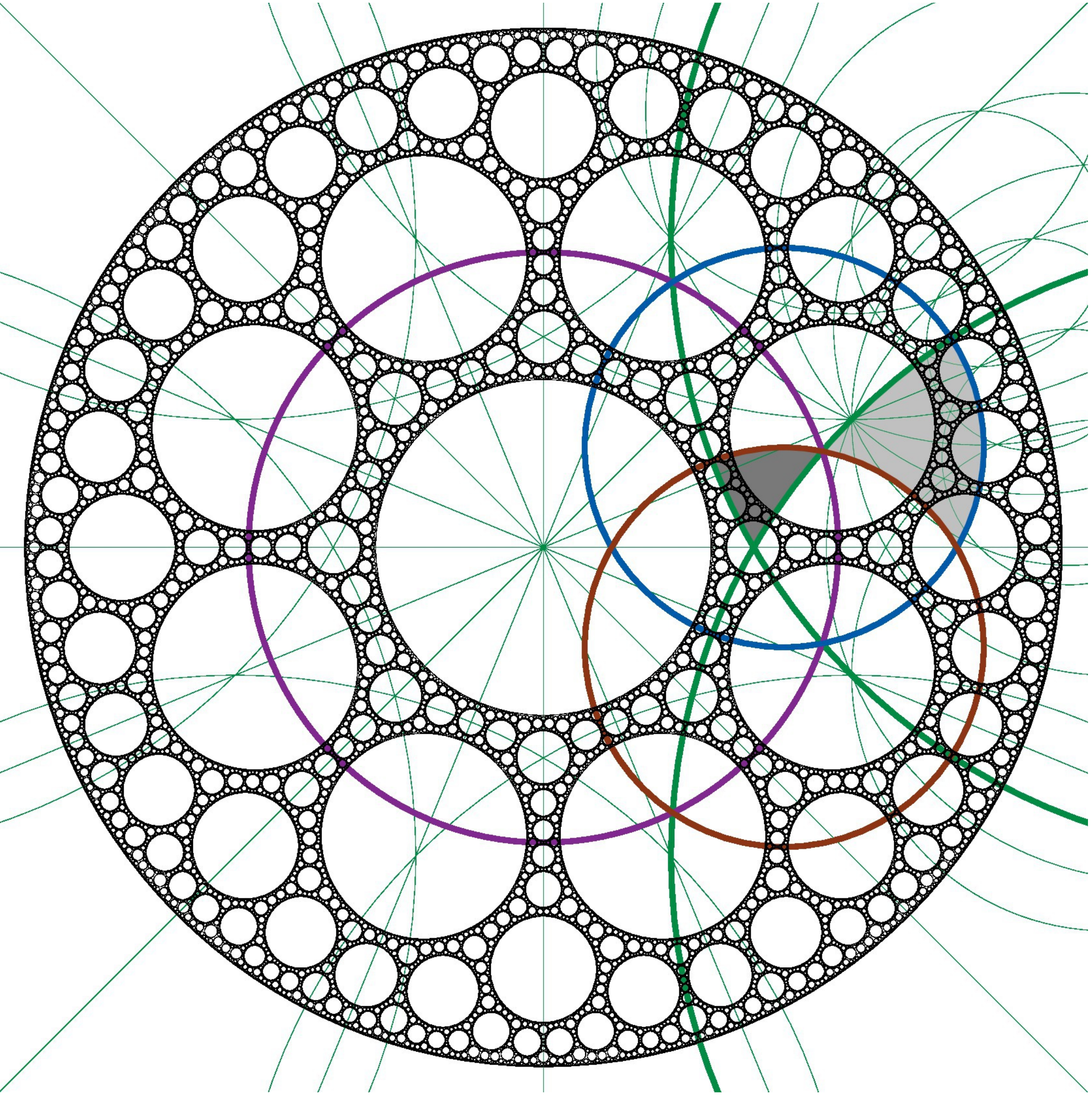

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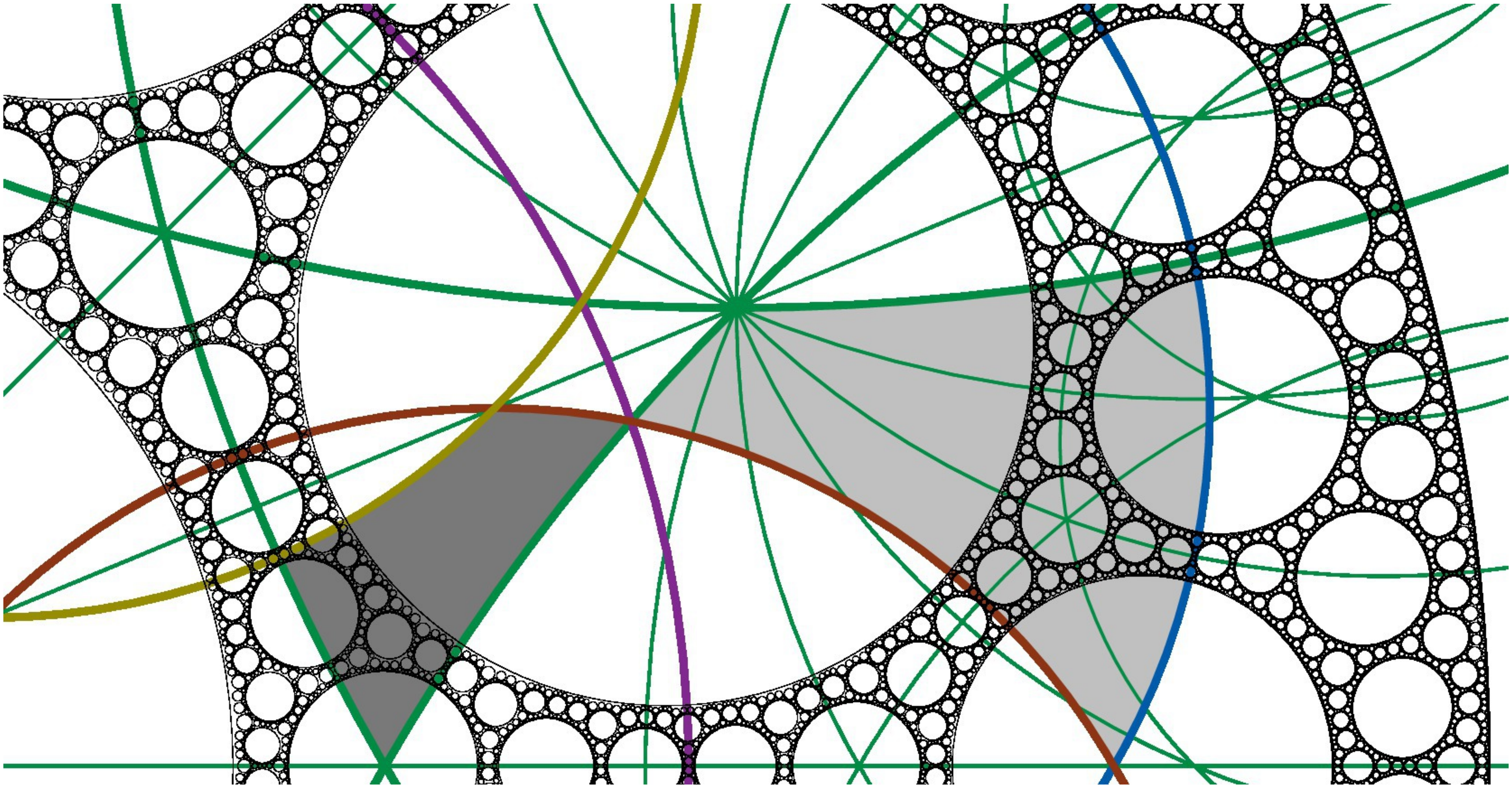

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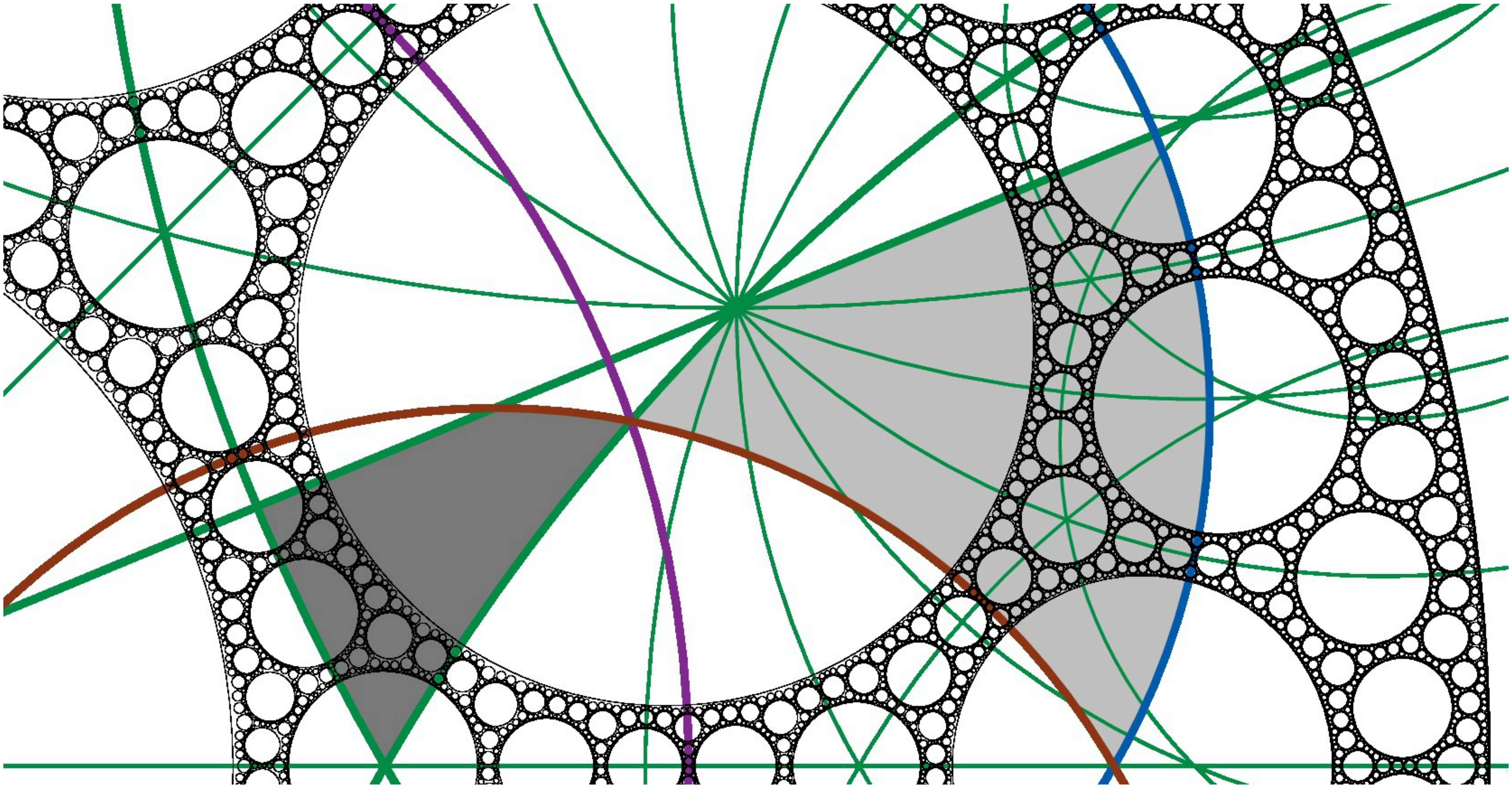
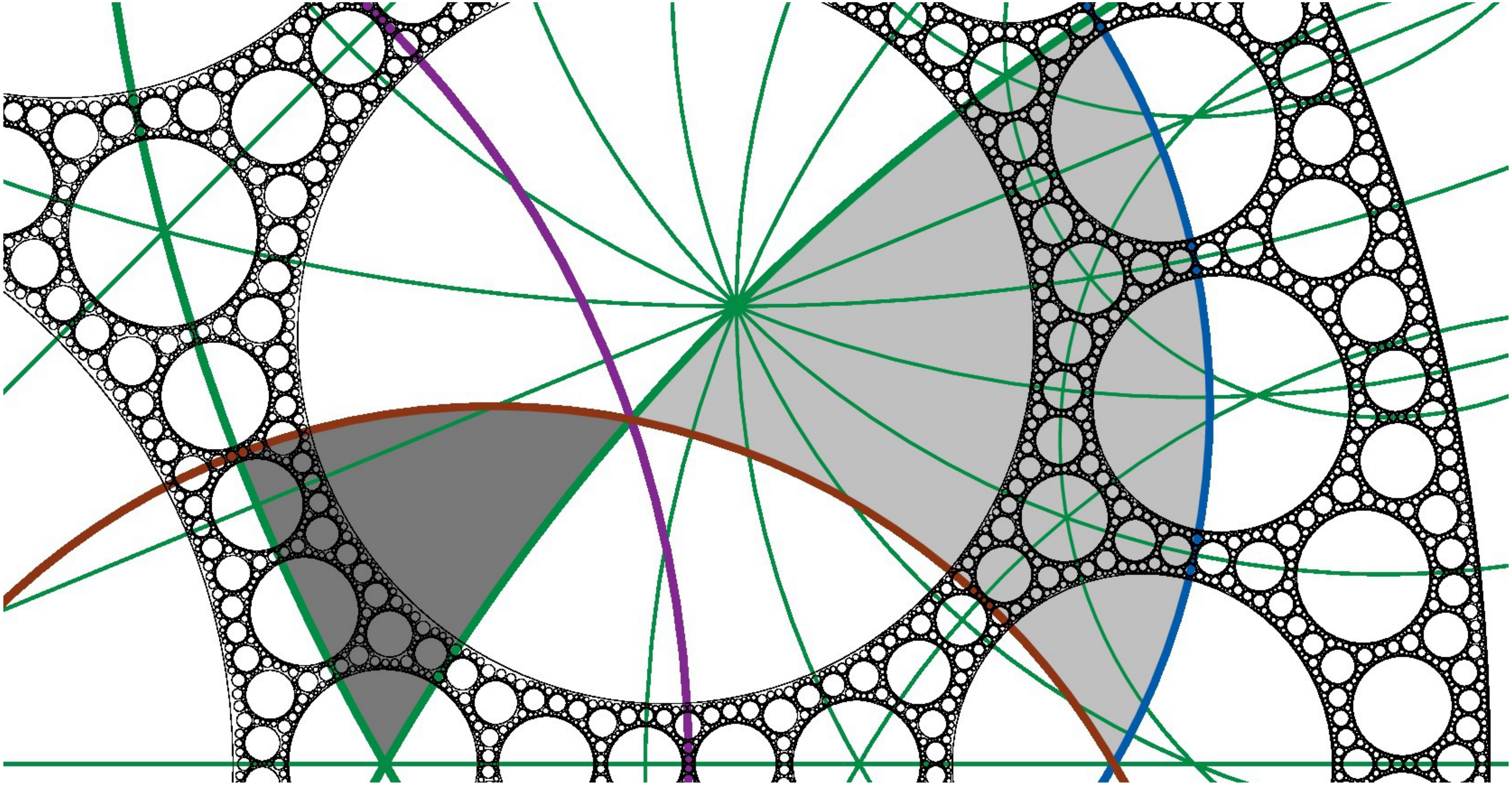


5 Ingredients of the proof of Weyl's asymptotics

- **A "self-similar" decomp.** ("fundamental domain" for the action $G \curvearrowright \partial_\infty G$)



- A **"self-similar" decomp.** ("fund. dom." for $G \curvearrowright \partial_\infty G$)
(requires concrete knowledge of $G \curvearrowright \partial_\infty G$; NOT extend easily)

- A version of the "2-dimensional" Nash inequality
($\rightsquigarrow$ the spectrum of Δ_{K_g} is discrete & $\exists p_t^{K_g}(x,y) \le c_g t^{-1}$)

- $\iota : K_g \hookrightarrow \mathbb{R}^2$ is $\mathcal{E}^g$-*harmonic* & $\Gamma_{\mathcal{E}^g}(\iota,\iota) = \nu^g$
($\rightsquigarrow \{\langle X_t^g, z\rangle\}_t$ slower than $\{B_t^{\mathbb{R}}\}_t \rightsquigarrow \mathbb{P}_x[\tau_{B(x,r)} \le t]$ small!)

$\rightsquigarrow$ can now run a Markov chain on the *Euclidean shapes of the cells* and apply Kesten's renewal thm [AOP '74].

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(requires concrete knowledge of $G \curvearrowright \partial_\infty G$; NOT extend easily)

- A version of the **"2-dimensional" Nash inequality**

(⇝ the spectrum of Δ_{K_g} is discrete & ${}^{\exists} p_t^{K_g}(x,y) \le c_g t^{-1}$)

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- (*cf.* Bonk '11) The circles in $\partial_\infty G$ are unif. rel. separated:
$^\forall j \neq k$, $\mathrm{dist}(C_j, C_k) \ge \delta_m \min\{\mathrm{rad}(C_j), \mathrm{rad}(C_k)\}$.

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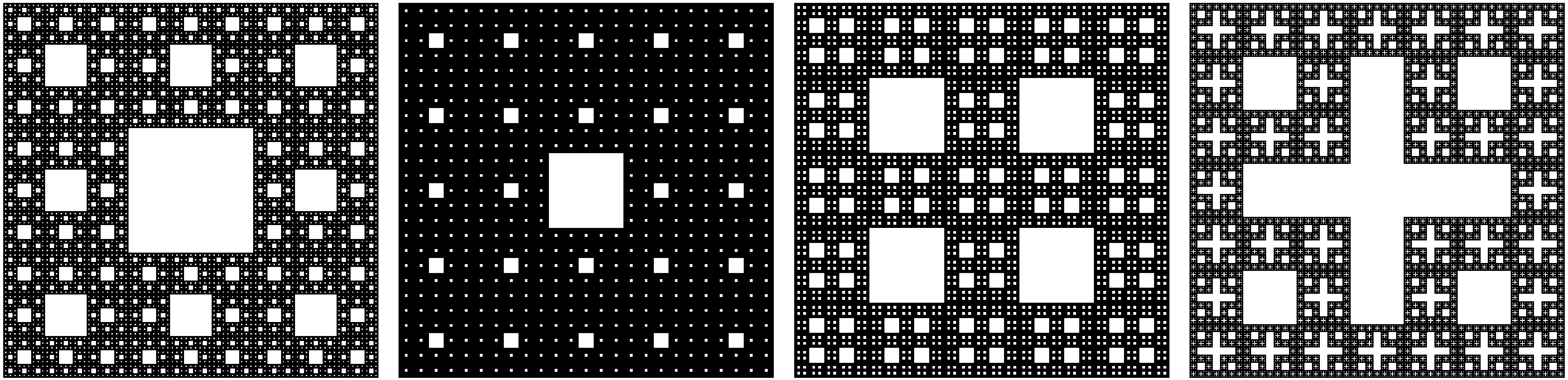

(generalized) self-similar SCs

Bonk '11: Each of them can be quasi-symmetrically mapped to a round SC in a unique way!

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Prf. To follow Kigami–Lapidus' method [CMP '93], we use Kesten's renewal thm for Markov chains [Ann. Prob. '74].

▷ $K_x\setminus V_0=\bigcup_{k=1}^{6}\bigcup_{l=1}^{\infty}K_{\phi_{k,l}(x)}$

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A key for Reminder estimate: Embedding in H^1!

Prop(K.). Let $f \in \mathrm{LIP}(\mathrm{Arc}(0,r,[0,\alpha]))$. Define $\mathcal{I}_{\mathrm{D}} f, \mathcal{I}_{\mathrm{N}} f : D(0,r,[0,\alpha]) \to \mathbb{R}$ by

$$\mathcal{I}_{\mathrm{D}} f(se^{i\theta}) := \left(1 - \frac{s}{r}\right) f(r) + \frac{s}{r} f(re^{i\theta}),$$

$$\mathcal{I}_{\mathrm{N}} f(se^{i\theta}) := \left(1 - \frac{s}{r}\right) \int_0^\alpha f(re^{i\theta}) \frac{d\theta}{\alpha} + \frac{s}{r} f(re^{i\theta}).$$

Then $\mathrm{Lip}(\mathcal{I}_{\mathrm{B}} f) \le 100 \mathrm{Lip}(f)$ and

$$\|\mathcal{I}_{\mathrm{B}} f\|_{L^2}^2 \asymp r \|f\|_{L^2}^2, \quad \|\nabla \mathcal{I}_{\mathrm{B}} f\|_{L^2}^2 \asymp r \|f'\|_{L^2}^2.$$

▷ $\nu^g := \sum_{C \subset_{\mathrm{arc}} K_g} \mathrm{rad}(C)\, d\mathrm{vol}_C$ (**NOT doubling!**)

▷ ${}^\forall u \in \mathrm{LIP}|_{K_g}$, $\mathcal{E}^g(u,u) := \sum_{C \subset_{\mathrm{arc}} K_g} \mathrm{rad}(C) \int_C |\nabla_C u|^2\, d\mathrm{vol}_C$ (*cf.* Osada '07)

A key for Reminder estimate: Embedding in H^1!

Prop(K.). Let $f \in \mathrm{LIP}(\mathrm{Arc}(0,r,[0,\alpha]))$. Define $\mathcal{I}_{\mathrm{D}} f, \mathcal{I}_{\mathrm{N}} f : D(0,r,[0,\alpha]) \to \mathbb{R}$ by

$$\mathcal{I}_{\mathrm{D}} f(se^{i\theta}) := \left(1 - \frac{s}{r}\right) f(r) + \frac{s}{r} f(re^{i\theta}),$$

$$\mathcal{I}_{\mathrm{N}} f(se^{i\theta}) := \left(1 - \frac{s}{r}\right) \int_0^\alpha f(re^{i\theta}) \frac{d\theta}{\alpha} + \frac{s}{r} f(re^{i\theta}).$$

Then $\mathrm{Lip}(\mathcal{I}_{\mathrm{B}} f) \leq 100 \mathrm{Lip}(f)$ and

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