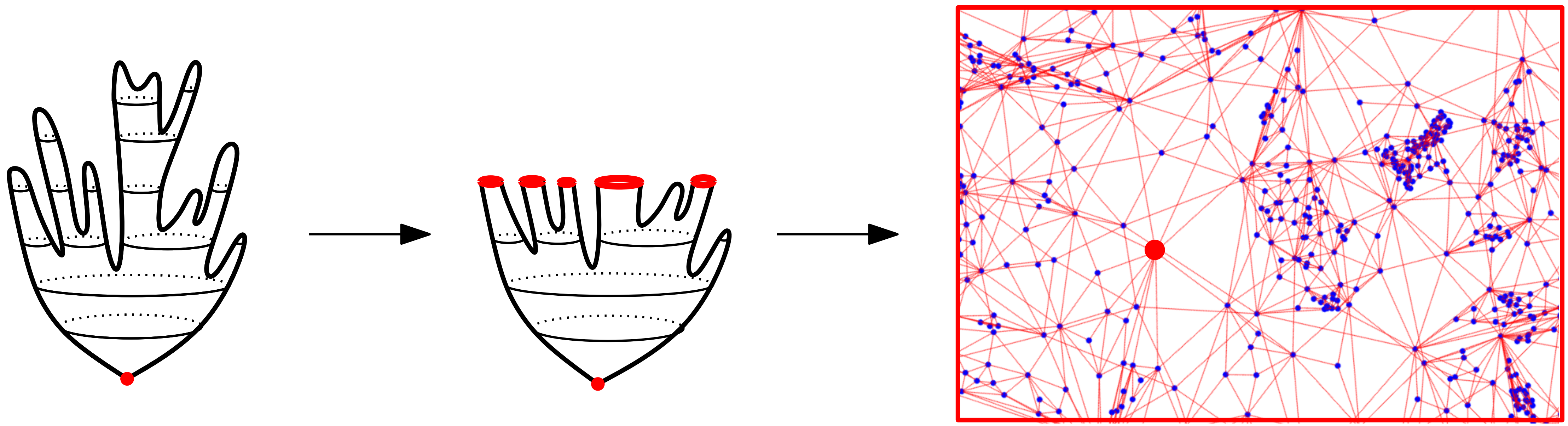
Random triangulations coupled with an Ising model

Laurent Ménard (Paris Nanterre)

joint work with **Marie Albenque** and **Gilles Schaeffer** (CNRS and LIX)



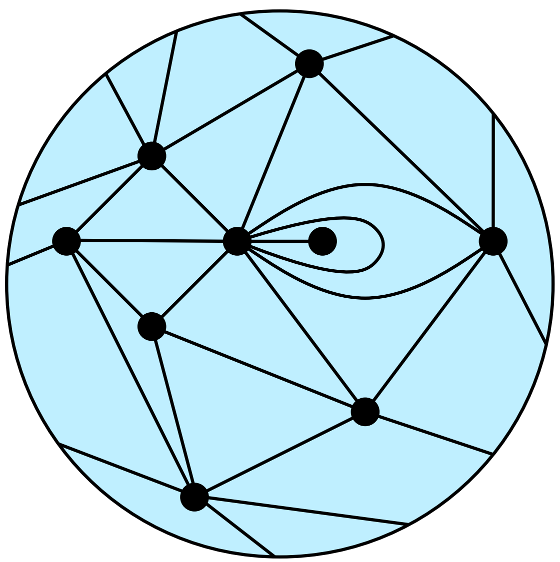
Nagoya, November 2018

Outline

1. Introduction: 2DQG and planar maps
2. Local weak topology
3. Uniform triangulations and the UIPT
4. Adding matter: Ising model
5. Combinatorics of triangulations with spins
6. Local limit of triangulations with spins

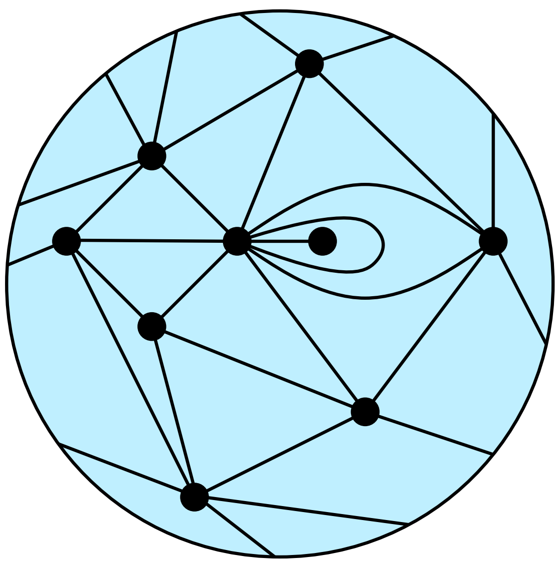
Planar Maps as discrete planar metric spaces

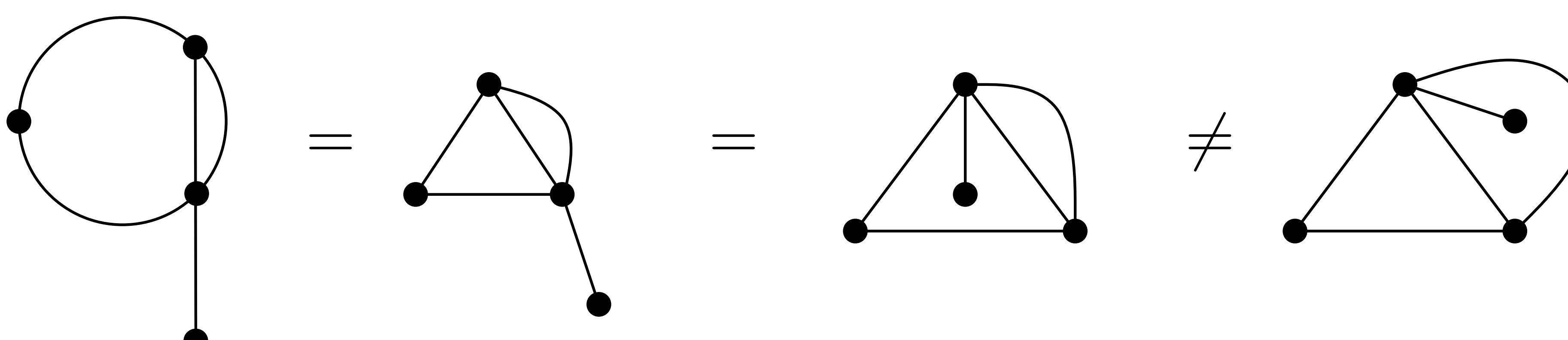
Definition:
A **planar map** is a proper embedding of a finite connected graph into the two-dimensional sphere (considered up to orientation-preserving homeomorphisms of the sphere).



Planar Maps as discrete planar metric spaces

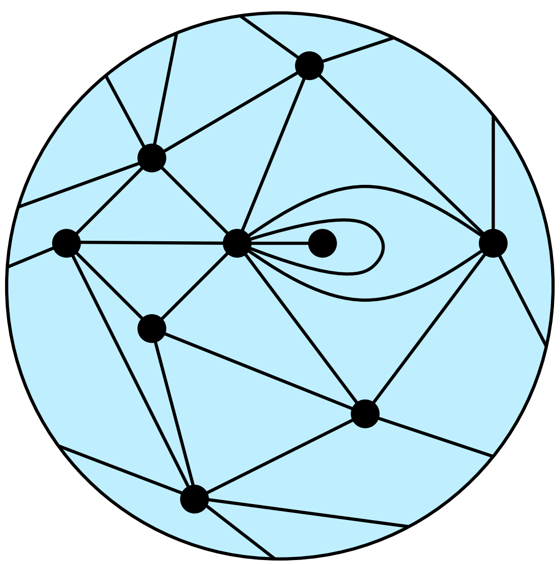
Definition:
A **planar map** is a proper embedding of a finite connected graph into the two-dimensional sphere (considered up to orientation-preserving homeomorphisms of the sphere).





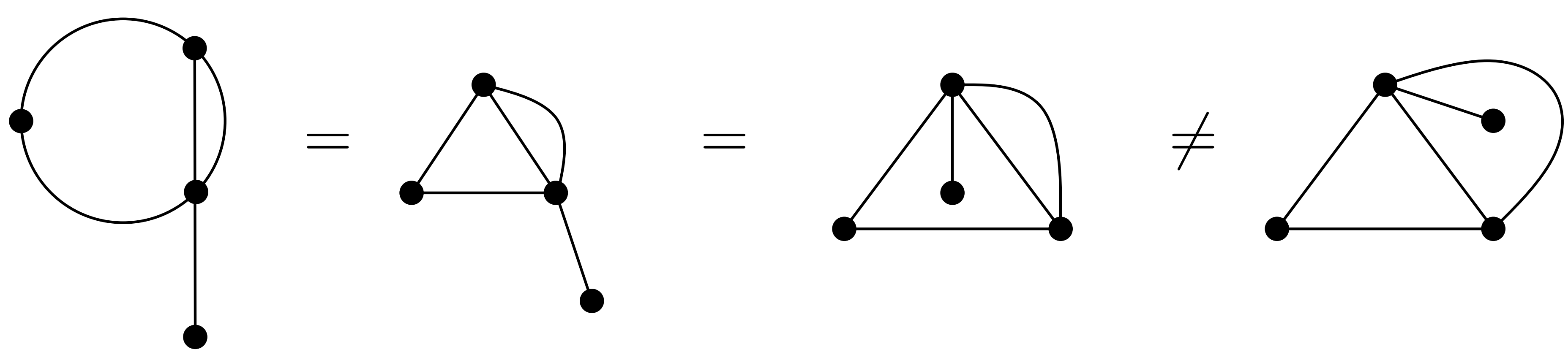
Planar Maps as discrete planar metric spaces

Definition:
A **planar map** is a proper embedding of a finite connected graph into the two-dimensional sphere (considered up to orientation-preserving homeomorphisms of the sphere).



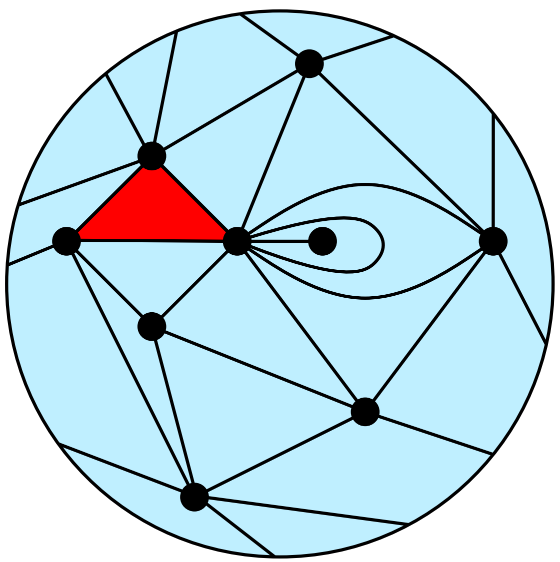
faces: connected components of the complement of edges

p-angulation: each face is bounded by p edges



Planar Maps as discrete planar metric spaces

Definition:
A **planar map** is a proper embedding of a finite connected graph into the two-dimensional sphere (considered up to orientation-preserving homeomorphisms of the sphere).

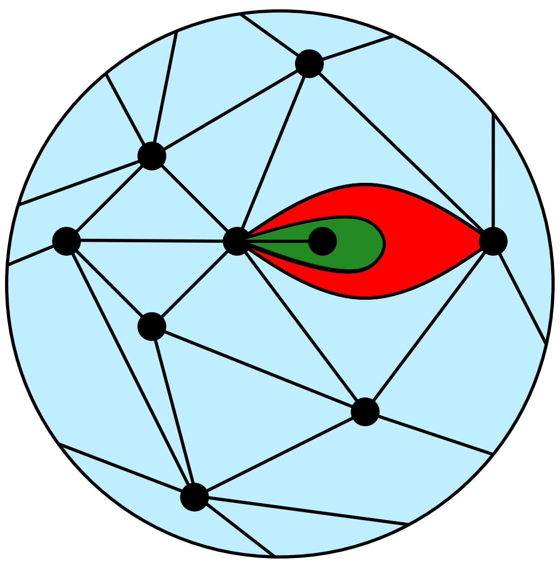


faces: connected components of the complement of edges

p**-angulation:** each face is bounded by p edges

Planar Maps as discrete planar metric spaces

Definition:
A **planar map** is a proper embedding of a finite connected graph into the two-dimensional sphere (considered up to orientation-preserving homeomorphisms of the sphere).

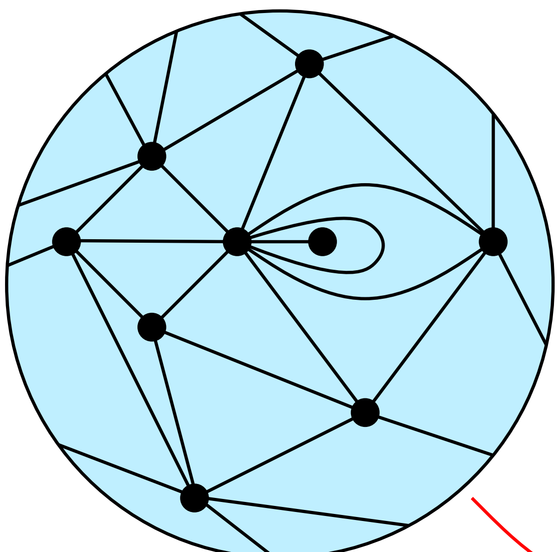


faces: connected components of the complement of edges

p-angulation: each face is bounded by p edges

Planar Maps as discrete planar metric spaces

Definition:
A **planar map** is a proper embedding of a finite connected graph into the two-dimensional sphere (considered up to orientation-preserving homeomorphisms of the sphere).



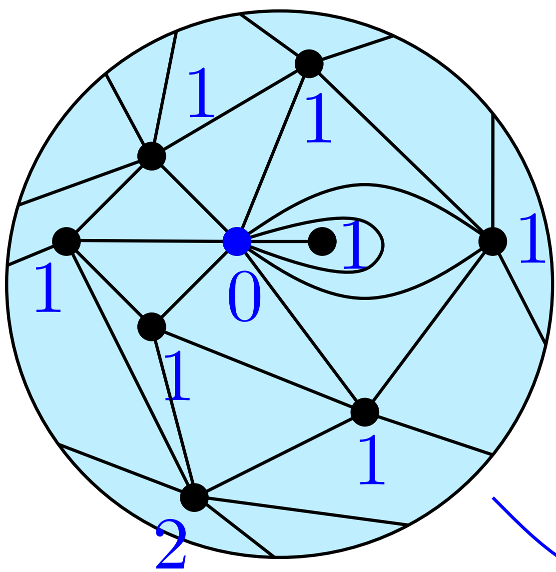
faces: connected components of the complement of edges

p-**angulation:** each face is bounded by p edges

This is a triangulation

Planar Maps as discrete planar metric spaces

Definition:
A **planar map** is a proper embedding of a finite connected graph into the two-dimensional sphere (considered up to orientation-preserving homeomorphisms of the sphere).



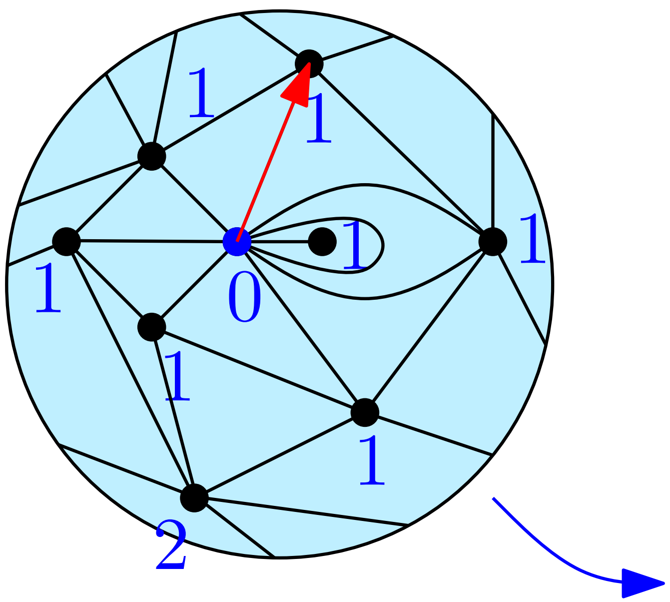
In blue, distances from •

M Planar Map:

- $V(M) :=$ set of vertices of M
- $d_{gr} :=$ graph distance on $V(M)$
- $(V(M), d_{gr})$ is a (finite) metric space

Planar Maps as discrete planar metric spaces

Definition:
A **planar map** is a proper embedding of a finite connected graph into the two-dimensional sphere (considered up to orientation-preserving homeomorphisms of the sphere).



In blue, distances from •

M Planar Map:

- $V(M) :=$ set of vertices of M
- $d_{gr} :=$ graph distance on $V(M)$
- $(V(M), d_{gr})$ is a (finite) metric space

Rooted map: mark an oriented edge of the map →

"Classical" large random triangulations

Euler relation in a triangulation: number of edges / vertices / faces linked

Take a triangulation of size n uniformly at random. What does it look like if n is large ?

Two points of view: global/local, continuous/discrete

"Classical" large random triangulations

Euler relation in a triangulation: number of edges / vertices / faces linked

Take a triangulation of size n uniformly at random. What does it look like if n is large ?

Two points of view: global/local, continuous/discrete

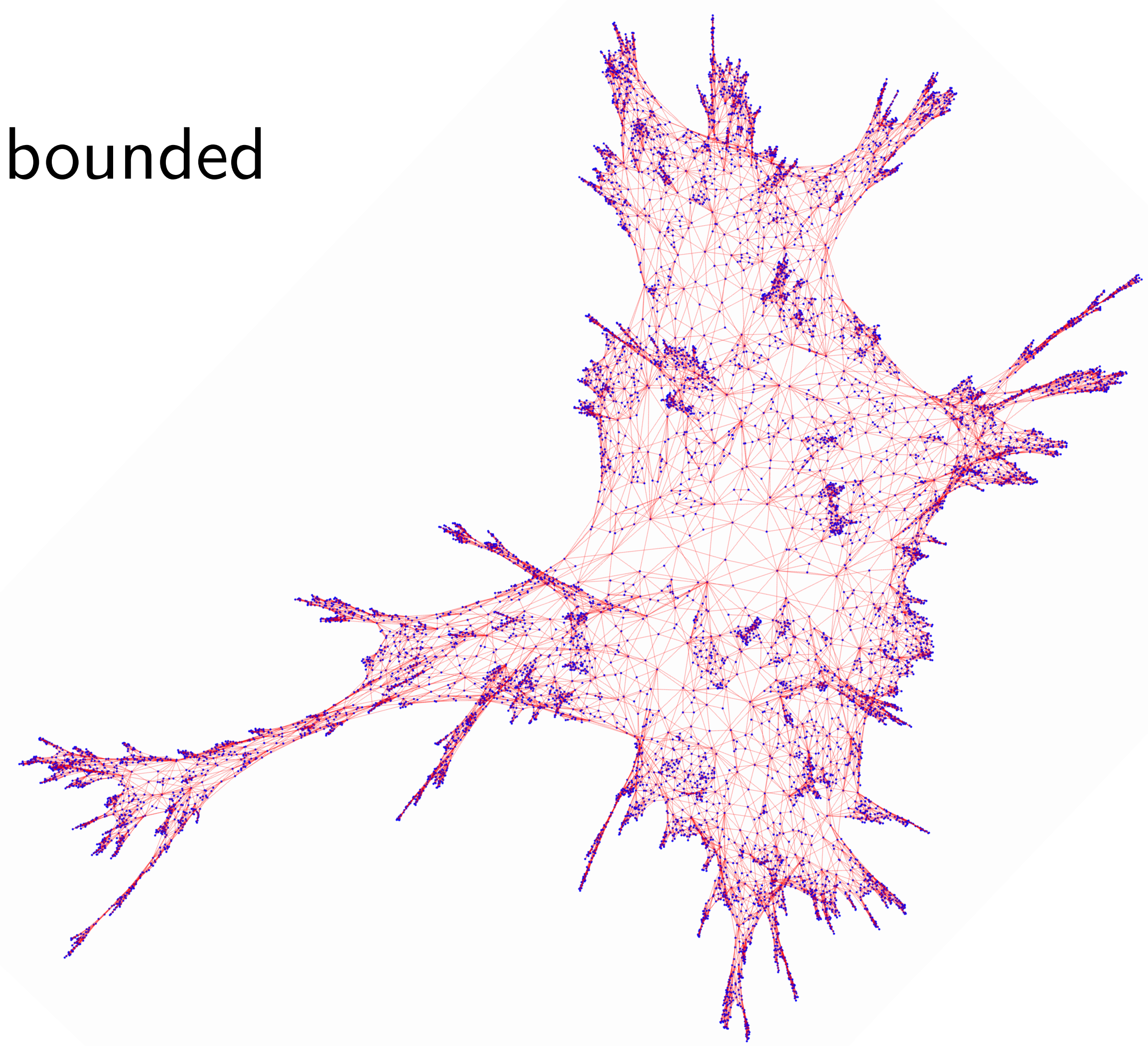
Global :

Rescale distances to keep diameter bounded

[Le Gall 13, Miermont 13]:

converges to the **Brownian map**.

- Gromov-Hausdorff topology
- Continuous metric space
- Homeomorphic to the sphere
- Hausdorff dimension 4
- **Universality**



"Classical" large random triangulations

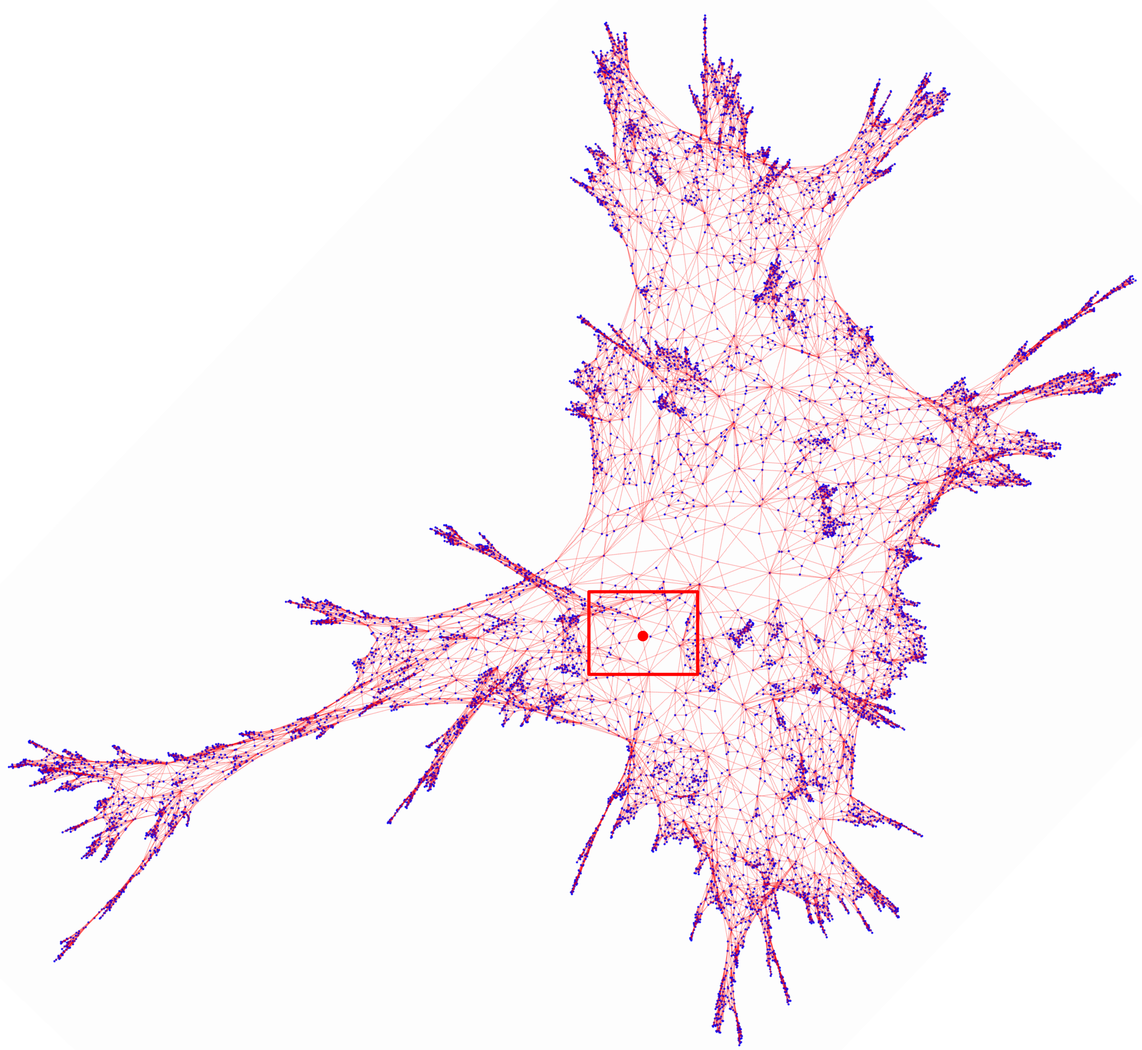
Euler relation in a triangulation: number of edges / vertices / faces linked

Take a triangulation of size n uniformly at random. What does it look like if n is large ?

Two points of view: global/local, continuous/discrete

Local :

Don't rescale distances and look at neighborhoods of the root



"Classical" large random triangulations

Euler relation in a triangulation: number of edges / vertices / faces linked

Take a triangulation of size n uniformly at random. What does it look like if n is large ?

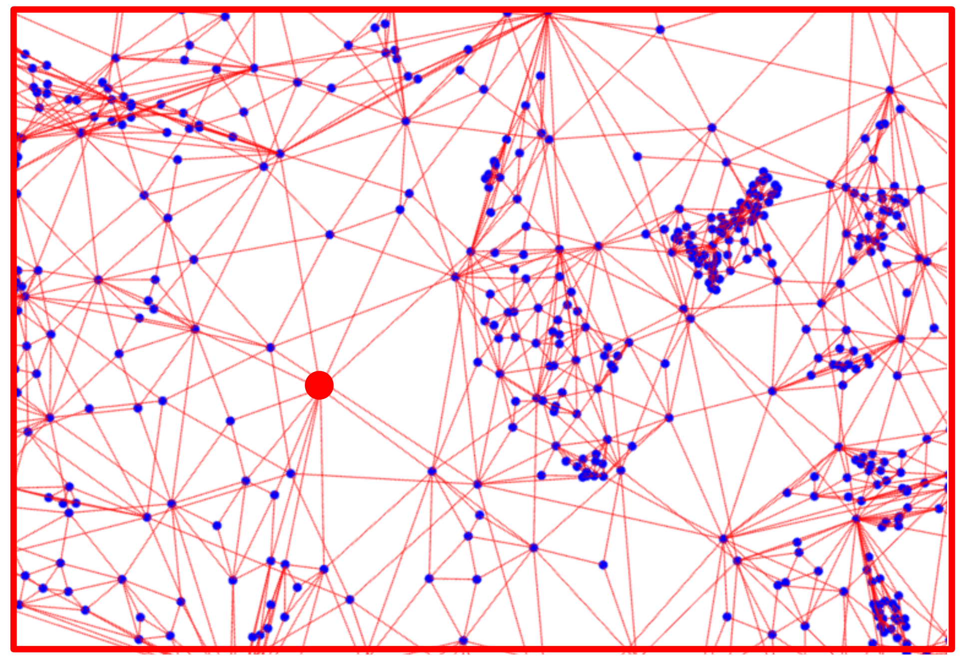
Two points of view: global/local, continuous/discrete

Local :

Don't rescale distances and look at neighborhoods of the root

[Angel – Schramm 03, Krikun 05]:

Converges to the **Uniform Infinite Planar Triangulation**



- Local topology
- Metric balls of radius R grow like R^4
- **"Universality"** of the exponent 4.

Local Topology

Local Topology for planar maps

$$\mathcal{M}_f := \{\text{finite rooted planar maps}\}.$$

Definition:

The **local topology** on $\mathcal{M}_f$ is induced by the distance:

$$d_{loc}(m, m') := \left(1 + \max\{r \geq 0 : B_r(m) = B_r(m')\}\right)^{-1}$$

where $B_r(m)$ is the graph made of all the vertices and edges of m which are within distance r from the root.

Local Topology for planar maps

$$\mathcal{M}_f := \{\text{finite rooted planar maps}\}.$$

Definition:
The **local topology** on $\mathcal{M}_f$ is induced by the distance:

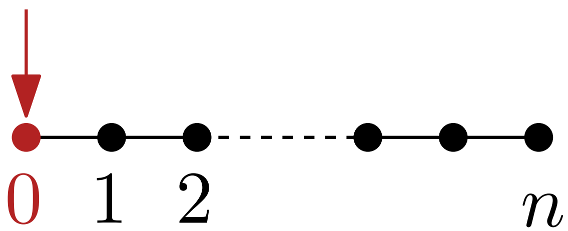
$$d_{loc}(m, m') := \left(1 + \max\{r \geq 0 : B_r(m) = B_r(m')\}\right)^{-1}$$

where $B_r(m)$ is the graph made of all the vertices and edges of m which are within distance r from the root.

- $(\mathcal{M}, d_{loc})$: closure of $(\mathcal{M}_f, d_{loc})$. It is a **Polish** space (complete and separable).

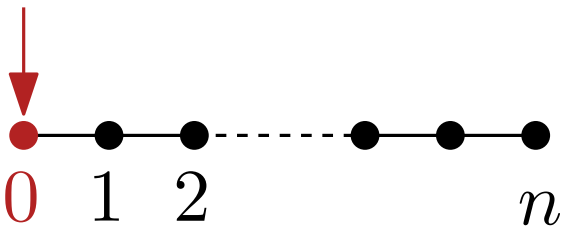
- $\mathcal{M}_\infty := \mathcal{M} \setminus \mathcal{M}_f$ set of infinite planar maps.

Local convergence: simple examples



Root $= 0$

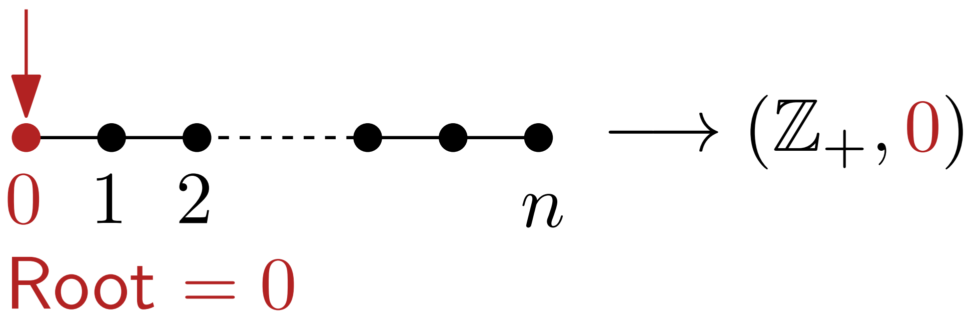
Local convergence: simple examples

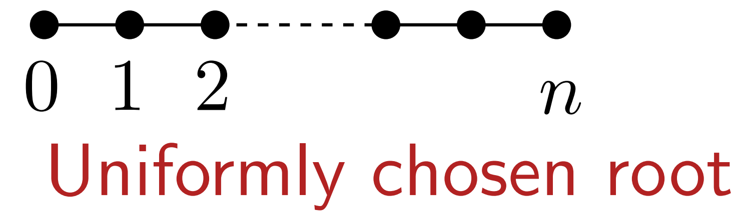


$\longrightarrow (\mathbb{Z}_+, 0)$

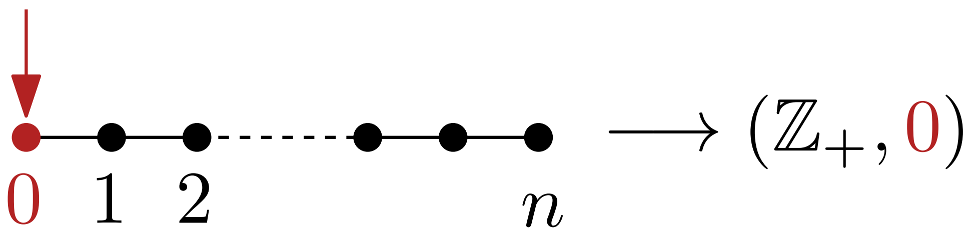
Root $= 0$

Local convergence: simple examples

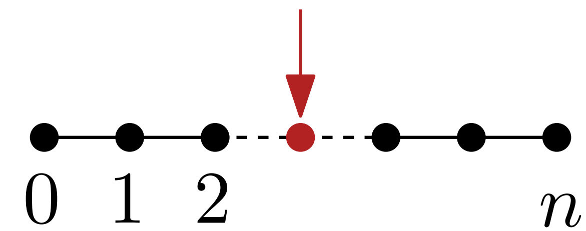




Local convergence: simple examples

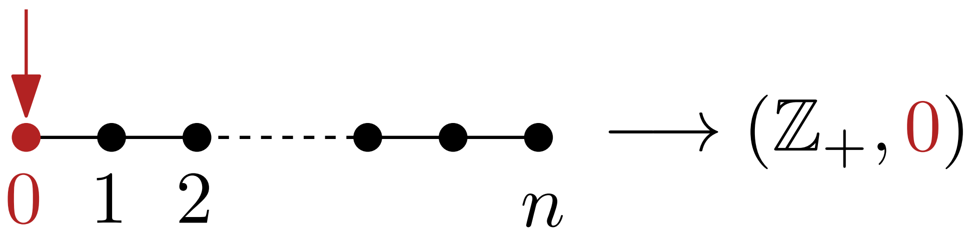


Root = 0

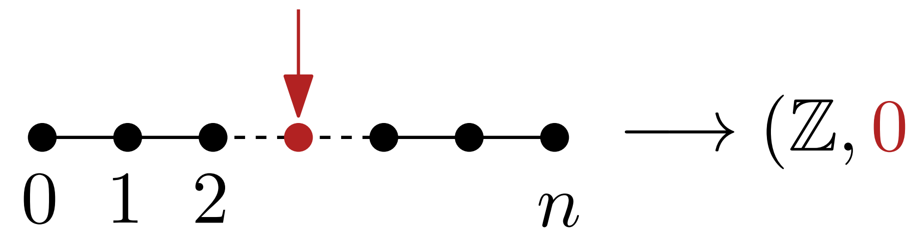


Uniformly chosen root

Local convergence: simple examples

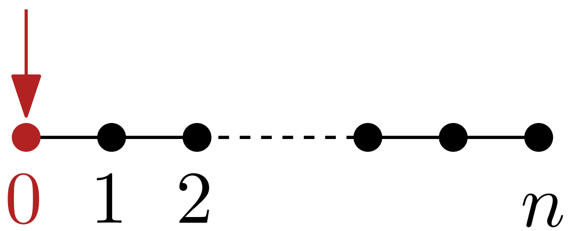


Root $= 0$



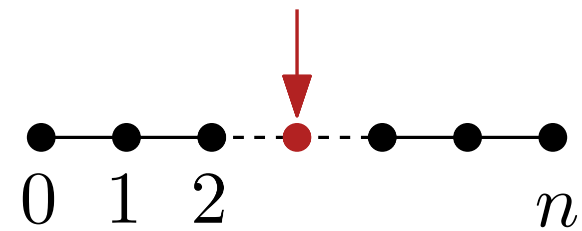
Uniformly chosen root

Local convergence: simple examples



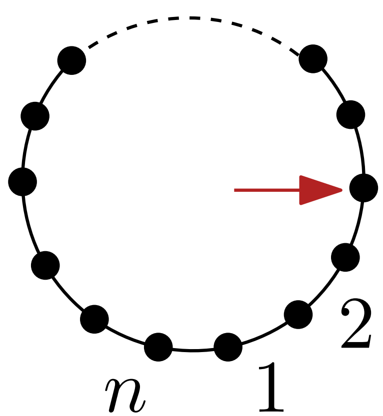
$\longrightarrow (\mathbb{Z}_+, 0)$

Root $= 0$



$\longrightarrow (\mathbb{Z}, 0)$

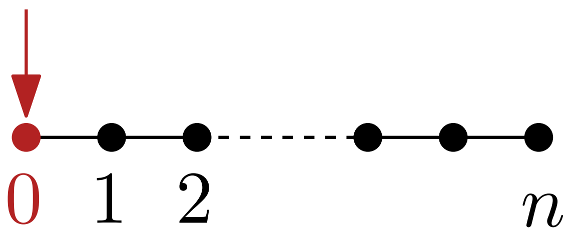
Uniformly chosen root



$\longrightarrow (\mathbb{Z}, 0)$

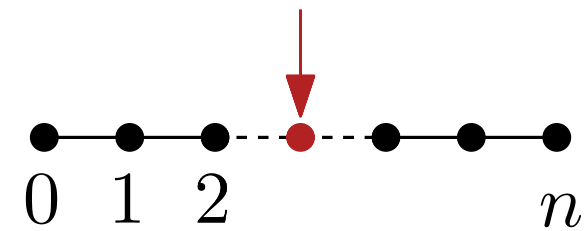
Root does not matter

Local convergence: simple examples



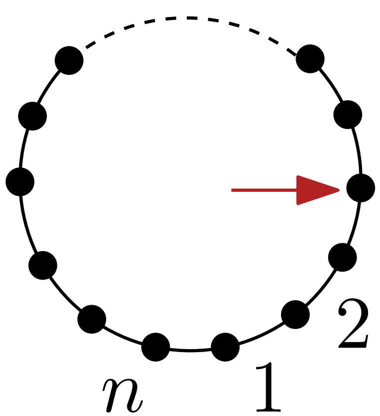
$\longrightarrow (\mathbb{Z}_+, 0)$

Root $= 0$



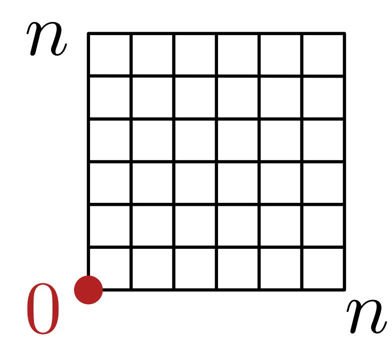
$\longrightarrow (\mathbb{Z}, 0)$

Uniformly chosen root



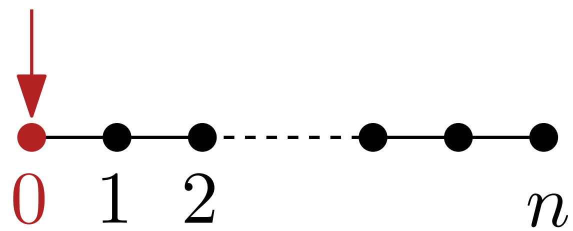
$\longrightarrow (\mathbb{Z}, 0)$

Root does not matter



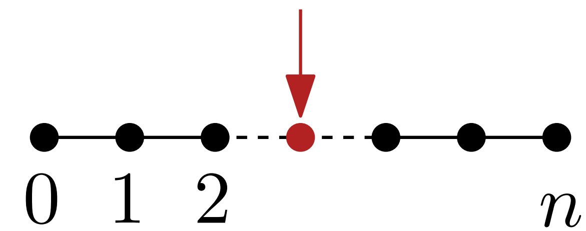
$\longrightarrow (\mathbb{Z}_+^2, 0)$

Local convergence: simple examples



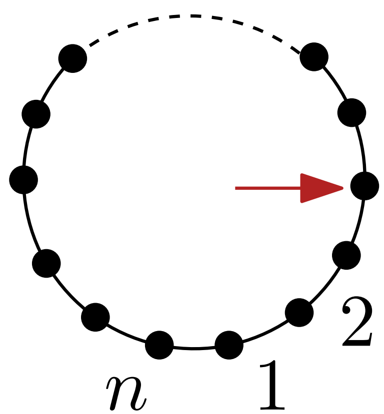
$\longrightarrow (\mathbb{Z}_+, 0)$

Root $= 0$



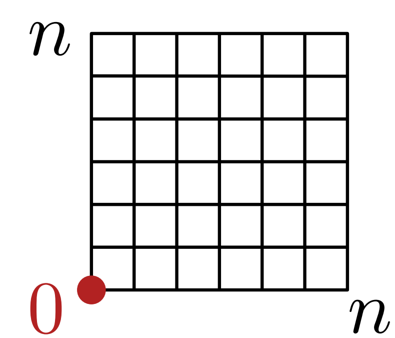
$\longrightarrow (\mathbb{Z}, 0)$

Uniformly chosen root

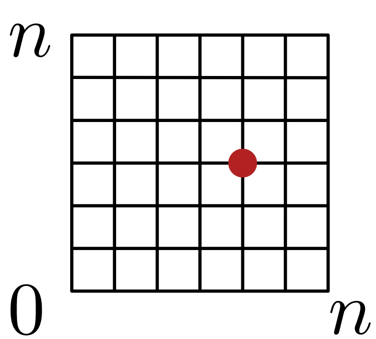


$\longrightarrow (\mathbb{Z}, 0)$

Root does not matter



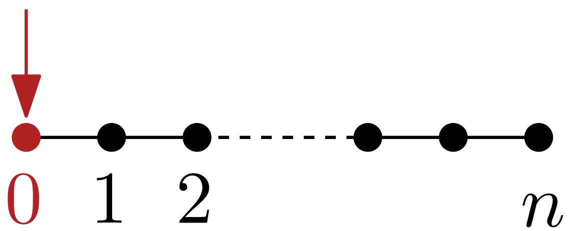
$\longrightarrow (\mathbb{Z}_+^2, 0)$



$\longrightarrow (\mathbb{Z}^2, 0)$

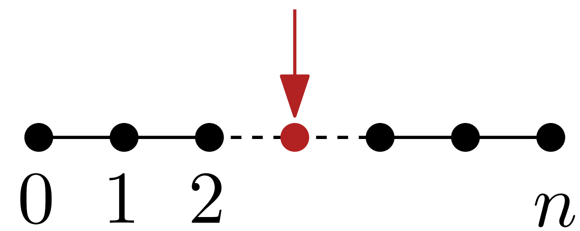
Uniformly chosen root

Local convergence: simple examples



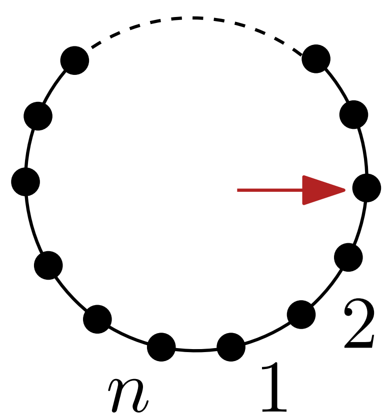
$\longrightarrow (\mathbb{Z}_+, 0)$

Root $= 0$



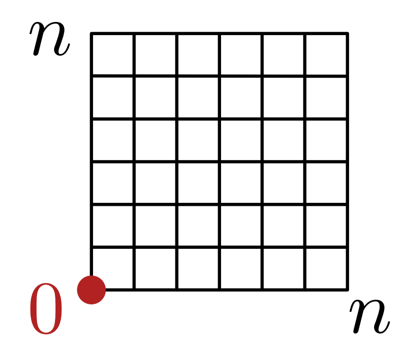
$\longrightarrow (\mathbb{Z}, 0)$

Uniformly chosen root

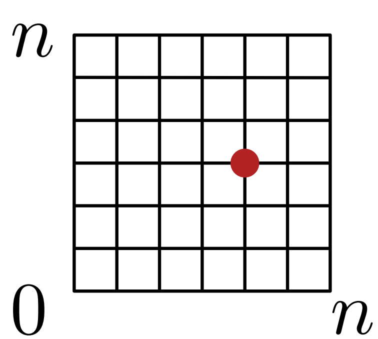


$\longrightarrow (\mathbb{Z}, 0)$

Root does not matter



$\longrightarrow (\mathbb{Z}_+^2, 0)$



$\longrightarrow (\mathbb{Z}^2, 0)$

Uniformly chosen root

Triangulation with n faces chosen uniformly at random $\longrightarrow$?

Local topology for triangulations

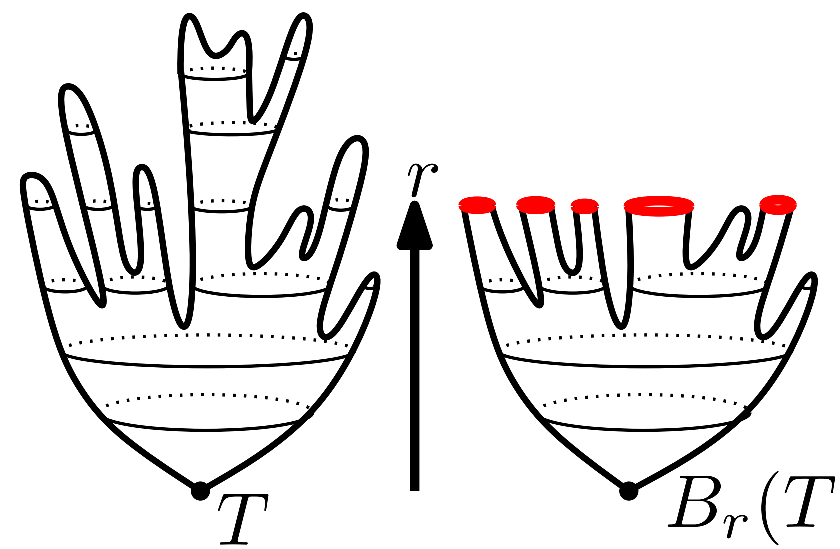
$$\mathcal{T}_f := \{\text{finite rooted planar triangulations}\}.$$

Definition:

The **local topology** on $\mathcal{T}_f$ is induced by the distance:

$$d_{loc}(T,T') := \left(1 + \max\{r \geq 0 : B_r(T) = B_r(T')\}\right)^{-1}$$

where $B_r(T)$ is the submap (with spins) of T composed by the **faces** of T with a vertex at distance $< r$ from the root.



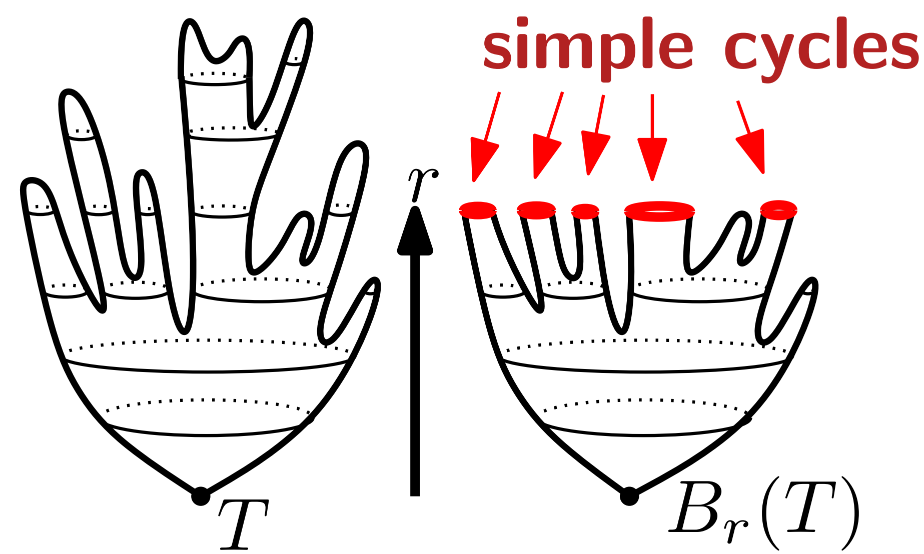
Local topology for triangulations

$$\mathcal{T}_f := \{\text{finite rooted planar triangulations}\}.$$

Definition:
The **local topology** on $\mathcal{T}_f$ is induced by the distance:

$$d_{loc}(T, T') := \left(1 + \max\{r \geq 0 : B_r(T) = B_r(T')\}\right)^{-1}$$

where $B_r(T)$ is the submap (with spins) of T composed by the **faces** of T with a vertex at distance $< r$ from the root.



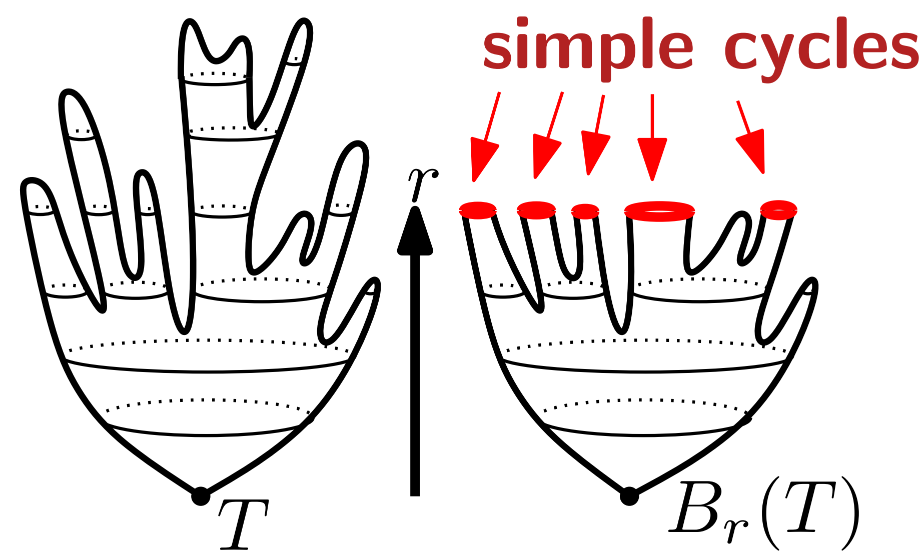
Local topology for triangulations

$$\mathcal{T}_f := \{\text{finite rooted planar triangulations}\}.$$

Definition:
The **local topology** on $\mathcal{T}_f$ is induced by the distance:

$$d_{loc}(T, T') := (1 + \max\{r \geq 0 : B_r(T) = B_r(T')\})^{-1}$$

where $B_r(T)$ is the submap (with spins) of T composed by the **faces** of T with a vertex at distance $< r$ from the root.



- $(\mathcal{T}, d_{loc})$: closure of $(\mathcal{T}_f, d_{loc})$. It is a **Polish** space.
- $\mathcal{T}_\infty := \mathcal{T} \setminus \mathcal{T}_f$ set of **infinite** planar triangulations.

Weak convergence for the local topology

Portemanteau theorem + Levy – Prokhorov metric:

A sequence of measures measures (P_n) on $\mathcal{T}_f$ converge weakly to a measure P on $\mathcal{T}_\infty$ if:

1. For every $r > 0$ and every possible r-ball Δ

$$P_n\left(\{(T,v) \in \mathcal{T}_f \,:\, B_r(T,v) = \Delta\} \right) \underset{n\to\infty}{\longrightarrow} P\left(\{T \in \mathcal{T}_\infty \,:\, B_r(T) = \Delta\} \right).$$

Weak convergence for the local topology

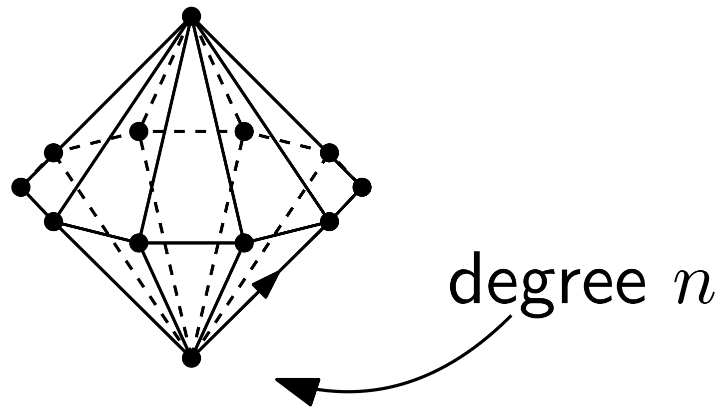
Portemanteau theorem + Levy – Prokhorov metric:
A sequence of measures measures (P_n) on $\mathcal{T}_f$ converge weakly to a measure P on $\mathcal{T}_\infty$ if:

1. For every $r > 0$ and every possible r-ball Δ

$$P_n\Big(\{(T,v) \in \mathcal{T}_f \,:\, B_r(T,v) = \Delta\} \Big) \underset{n\to\infty}{\longrightarrow} P\Big(\{T \in \mathcal{T}_\infty \,:\, B_r(T) = \Delta\} \Big).$$

Problem: not sufficient since the space $(\mathcal{T}, d_{loc})$ is **not compact**!

Ex:



Weak convergence for the local topology

Portemanteau theorem + Levy – Prokhorov metric:

A sequence of measures measures (P_n) on $\mathcal{T}_f$ converge weakly to a measure P on $\mathcal{T}_\infty$ if:

1. For every $r > 0$ and every possible r-ball Δ

$$P_n\Big(\{(T,v) \in \mathcal{T}_f \,:\, B_r(T,v) = \Delta\} \Big) \underset{n\to\infty}{\longrightarrow} P\Big(\{T \in \mathcal{T}_\infty \,:\, B_r(T) = \Delta\} \Big).$$

2. No loss of mass at the limit: Tightness of (P_n), **or** the measure P defined by the limits in 1. is a probability measure.

- Vertex degrees are tight (at finite distance from the root)
- $$\forall r > 0, \qquad \sum_{r-\text{balls}\,\Delta} P\Big(\{T \in \mathcal{T}_\infty \,:\, B_r(T) = \Delta\} \Big) = 1.$$

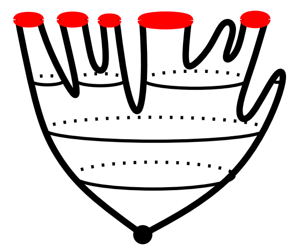
Uniform triangulations and the UIPT

Uniform triangulations probabilities

Denote $\mathbb{P}_n$ the uniform measure on triangulations with n vertices.

We want to compute, for every r and every fixed possible r-ball:

$$\mathbb{P}_n\left(B_r(\mathbf{T}) = \right.$$



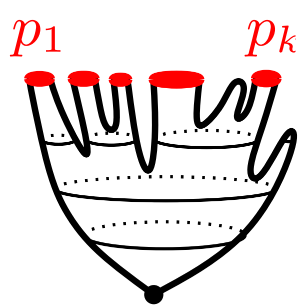
$$\left. \right)$$

Uniform triangulations probabilities

Denote $\mathbb{P}_n$ the uniform measure on triangulations with n vertices.

We want to compute, for every r and every fixed possible r-ball:

$$\mathbb{P}_n\Bigg(B_r(\mathbf{T}) =$$



$$\Bigg)$$

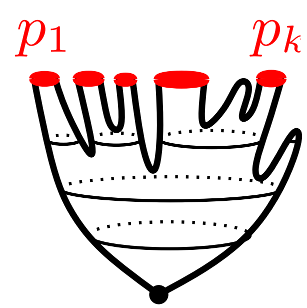
A fixed r-ball Δ has:

- $|\Delta|$ vertices.
- boundary lengths $(p_1, \ldots, p_k)$

Uniform triangulations probabilities

Denote $\mathbb{P}_n$ the uniform measure on triangulations with n vertices.

We want to compute, for every r and every fixed possible r-ball:

$\mathbb{P}_n\Big(B_r(\mathbf{T}) =$ 

$\Big)$

A fixed r-ball Δ has:
- $|\Delta|$ vertices.
- boundary lengths $(p_1, \dots, p_k)$

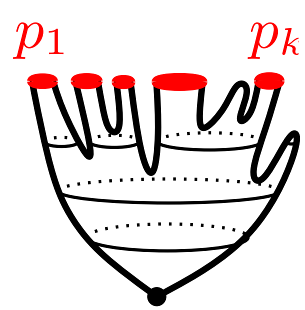
$t_{n,p} :=$ number of triangulations with boundary length p and n **inner** vertices

Uniform triangulations probabilities

Denote $\mathbb{P}_n$ the uniform measure on triangulations with n vertices.

We want to compute, for every r and every fixed possible r-ball:

$$\mathbb{P}_n\Bigg(B_r(\mathbf{T}) =$$



$$\Bigg) = \frac{\sum_{n_1+\cdots+n_k=n-|\Delta|} \prod_{i=1}^{k} t_{n_i,p_i}}{t_n}$$

A fixed r-ball Δ has:

- $|\Delta|$ vertices.
- boundary lengths $(p_1,\ldots,p_k)$

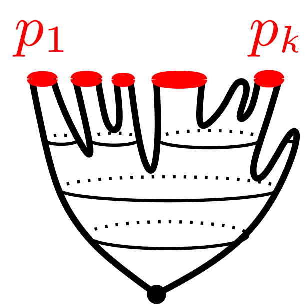
$t_{n,p} :=$ number of triangulations with boundary length p and n **inner** vertices

Uniform triangulations probabilities

Denote $\mathbb{P}_n$ the uniform measure on triangulations with n vertices.

We want to compute, for every r and every fixed possible r-ball:

$\mathbb{P}_n\Big(B_r(\mathbf{T}) =$



$\Big)$

$$= \frac{\sum_{n_1+\cdots+n_k=n-|\Delta|} \prod_{i=1}^{k} t_{n_i,p_i}}{t_n}$$

A fixed r-ball Δ has:

- $|\Delta|$ vertices.
- boundary lengths $(p_1, \ldots, p_k)$

$t_{n,p} :=$ number of triangulations with boundary length p and n **inner** vertices

?

Counting Maps: combinatorial toolbox

Theorem: [Tutte 60s]
The number of rooted triangulations with simple boundary of length p and with n inner vertices is

$$t_{n,p} := 4^{n-1} \frac{p\,(2p)!}{(p!)^2} \frac{(2p+3n-5)!!}{n!\,(2p+n-1)!!} \sim_{n\to\infty} C(p)(12\sqrt{3})^n n^{-5/2}.$$

Counting Maps: combinatorial toolbox

Theorem: [Tutte 60s]

The number of rooted triangulations with simple boundary of length p and with n inner vertices is

$$t_{n,p} := 4^{n-1}\frac{p\,(2p)!}{(p!)^2}\,\frac{(2p+3n-5)!!}{n!\,(2p+n-1)!!} \sim_{n\to\infty} C(p)(12\sqrt{3})^n n^{-5/2}.$$

Important Remark:

The asymptotic behavior $t_{n,p} \sim C(p)\,\rho^n n^{-5/2}$ is **universal**.

Counting Maps: combinatorial toolbox

Theorem: [Tutte 60s]
The number of rooted triangulations with simple boundary of length p and with n inner vertices is

$$t_{n,p} := 4^{n-1} \frac{p\,(2p)!}{(p!)^2} \frac{(2p+3n-5)!!}{n!\,(2p+n-1)!!} \sim_{n\to\infty} C(p)(12\sqrt{3})^n n^{-5/2}.$$

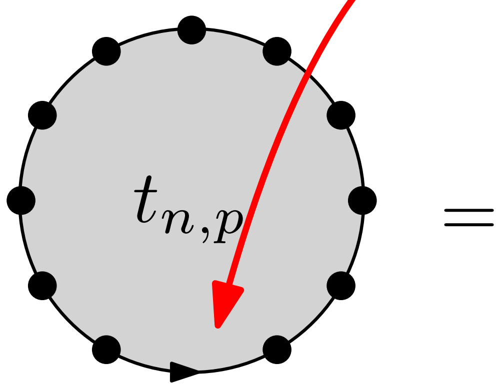
Main methods for counting maps:

- Generating functionology [Tutte 60s]:
  Encoding a recurrence relation via generating functions.
- Matrix integrals [Brézin-Itzykson-Parisi-Zuber 78]:
  Interpreting maps as the Feynman diagrams.
- Computation of characters [Goulden-Jackson]:
  Interpreting maps as products of permutations.
- Bijections with decorated trees [Cori-Vauquelin 81, Schaeffer 98].

Counting triangulations: one catalytic variable

Tutte / loop / Schwinger-Dyson / ... equations:

look at the face adjacent to the root edge (if any)

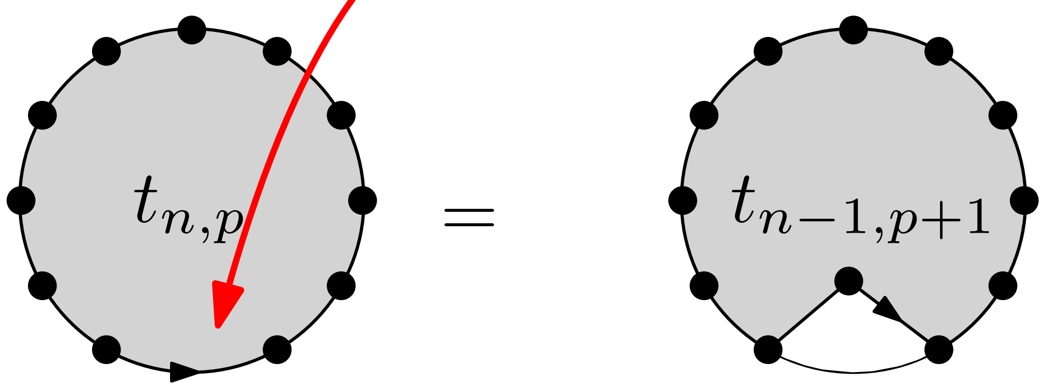


$=$

Counting triangulations: one catalytic variable

Tutte / loop / Schwinger-Dyson / ... equations:

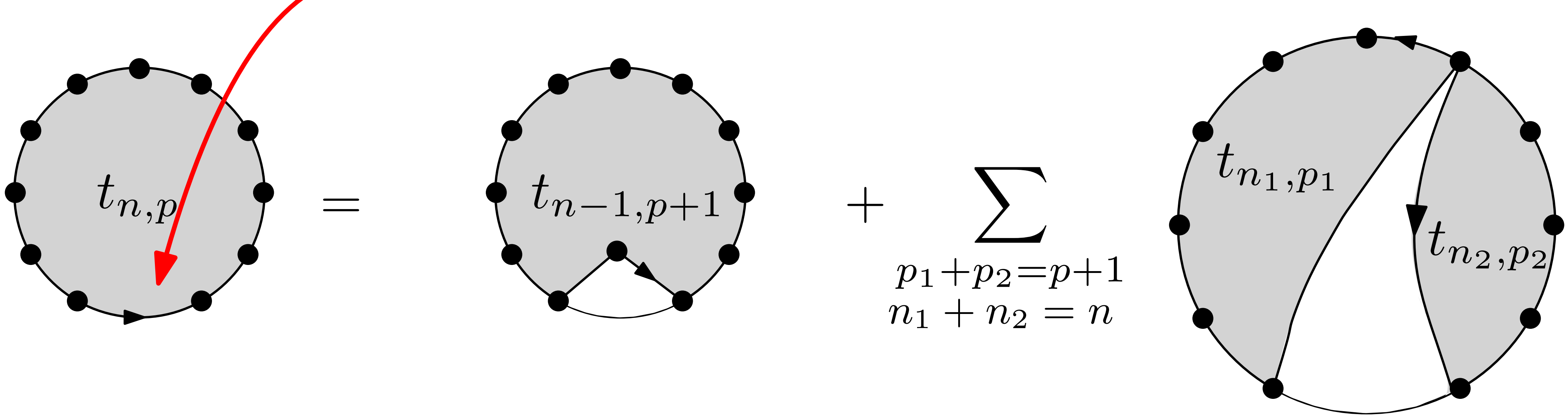
look at the face adjacent to the root edge (if any)



Counting triangulations: one catalytic variable

Tutte / loop / Schwinger-Dyson / ... equations:

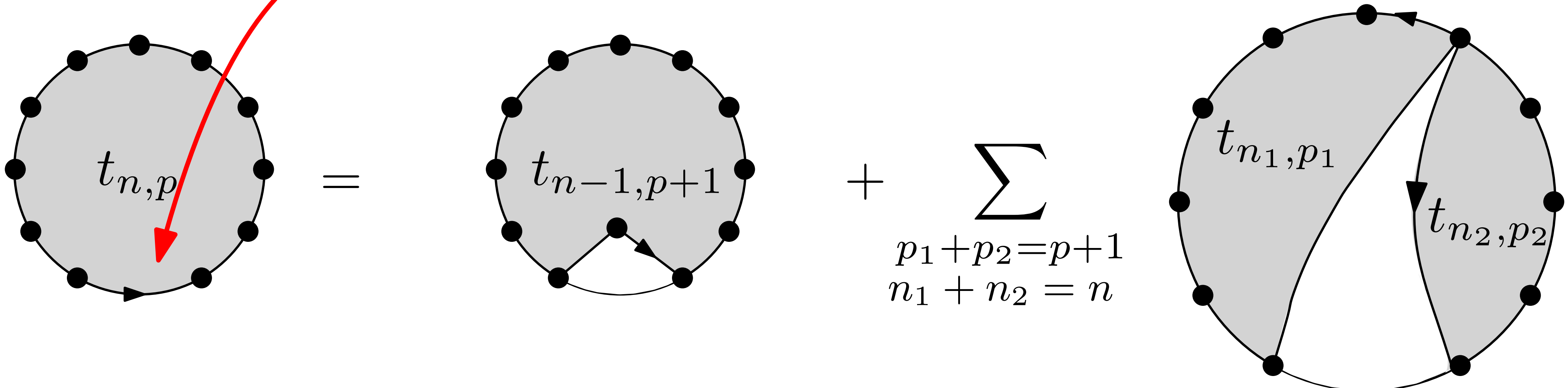
look at the face adjacent to the root edge (if any)



Counting triangulations: one catalytic variable

Tutte / loop / Schwinger-Dyson / ... equations:

look at the face adjacent to the root edge (if any)



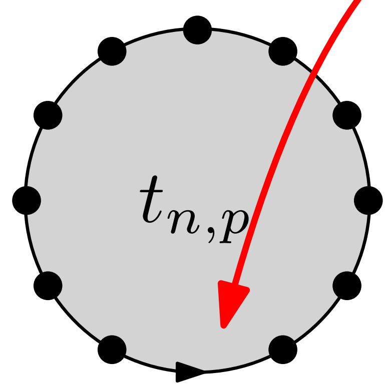
$$T(x,y) := \sum_{n\geq 0, p\geq 1} t_{n,p} x^n y^{p-1} = \sum_{p\geq 1} T_p(x) y^{p-1}$$

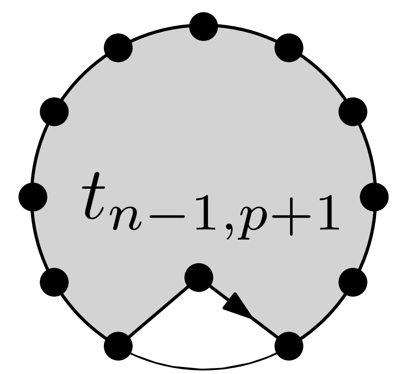
$$= y + \sum_{n\geq 1, p\geq 1} t_{n-1,p+1} x^n y^{p-1} + \sum_{n_1,n_2\geq 0, p_1,p_2\geq 1} t_{n_1,p_1} t_{n_2,p_2} x^{n_1+n_2} y^{p_1+p_2-2}$$

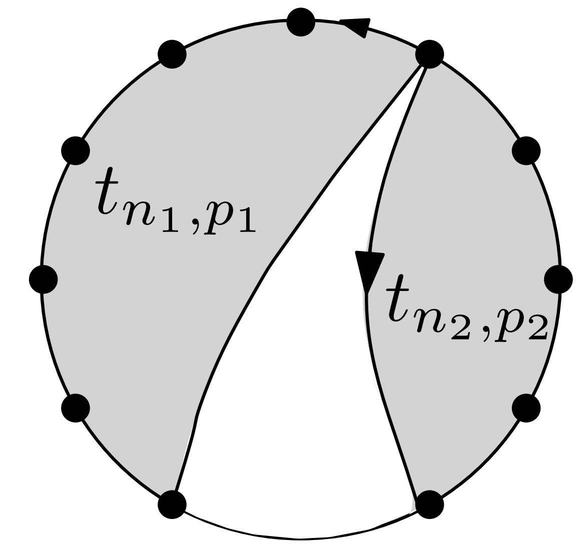
Counting triangulations: one catalytic variable

Tutte / loop / Schwinger-Dyson / ... equations:

look at the face adjacent to the root edge (if any)



$=$


$+ \sum_{\substack{p_1+p_2=p+1 \\ n_1+n_2=n}}$


$$T(x,y) := \sum_{n\geq 0, p\geq 1} t_{n,p} x^n y^{p-1} = \sum_{p\geq 1} T_p(x) y^{p-1}$$

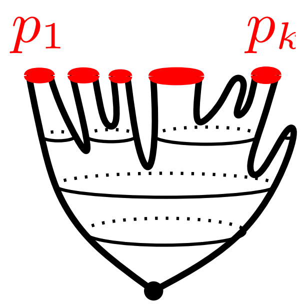
$$= y + \sum_{n\geq 1, p\geq 1} t_{n-1,p+1} x^n y^{p-1} + \sum_{n_1,n_2\geq 0, p_1,p_2\geq 1} t_{n_1,p_1} t_{n_2,p_2} x^{n_1+n_2} y^{p_1+p_2-2}$$

$$T(x,y) = y + \frac{x}{y}\left(T(x,y) - T(x,0)\right) + T(x,y)^2$$

y is a **catalytic** variable

Large triangulations: one endedness

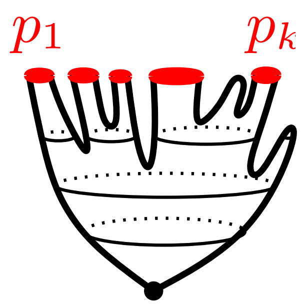
$$\mathbb{P}_n\Bigg(B_r(\mathbf{T}) =$$



$$\Bigg) = \frac{\displaystyle\sum_{n_1+\cdots+n_k=n-|\Delta|} \prod_{i=1}^{k} t_{n_i,p_i}}{t_{n,1}}$$

For fixed p, as $n \to \infty$, $t_{n,p} \sim C(p)\rho^{-n}\, n^{-\alpha}$ with $\alpha = \frac{5}{2}$

Large triangulations: one endedness

$$\mathbb{P}_n\Bigg(B_r(\mathbf{T}) = $$  $$\Bigg) = \frac{\sum_{n_1+\cdots+n_k=n-|\Delta|} \prod_{i=1}^{k} t_{n_i,p_i}}{t_{n,1}}$$

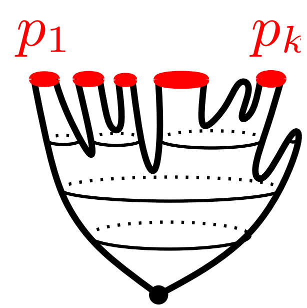
For fixed p, as $n \to \infty$, $t_{n,p} \sim C(p)\rho^{-n} n^{-\alpha}$ with $\alpha = \frac{5}{2}$

Since $\alpha > 2$: $$\sum_{\substack{n_1+\cdots+n_k=N \\ 2 \text{ of the } n_i\text{'s} > A}} \prod_{i=1}^{k} t_{n_i,p_i} \leq Cst \times \rho^{-N} N^{-\alpha} A^{1-\alpha}$$

meaning only one of the holes of $B_r(\mathbf{T})$ will be filled by an infinite triangulation.

Large triangulations: one endedness

$$\mathbb{P}_n\Bigg(B_r(\mathbf{T}) = $$



$$\Bigg) = \frac{\displaystyle\sum_{n_1+\cdots+n_k=n-|\Delta|} \prod_{i=1}^{k} t_{n_i,p_i}}{t_{n,1}}$$

For fixed p, as $n \to \infty$, $t_{n,p} \sim C(p)\rho^{-n} n^{-\alpha}$ with $\alpha = \frac{5}{2}$

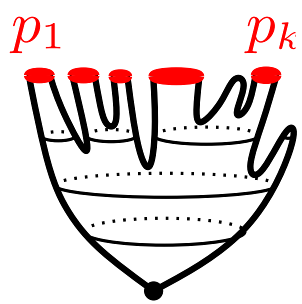
Since $\alpha > 2$: $$\sum_{\substack{n_1+\cdots+n_k=N \\ \text{2 of the } n_i\text{'s} > A}} \prod_{i=1}^{k} t_{n_i,p_i} \leq Cst \times \rho^{-N} N^{-\alpha} A^{1-\alpha}$$

meaning only one of the holes of $B_r(\mathbf{T})$ will be filled by an infinite triangulation.

one endedness property: if there is a limiting law for $\mathbb{P}_n$, it is supported on infinite triangulations with one end.

The Uniform Infinite Planar Triangulation

$$\mathbb{P}_n\left(B_r(\mathbf{T}) = \right.$$



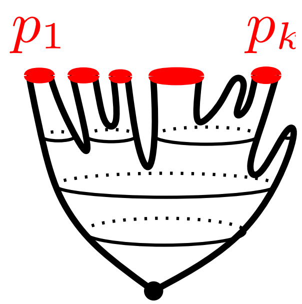
$$\left.\right) = \frac{\displaystyle\sum_{n_1+\cdots+n_k=n-|\Delta|} \prod_{i=1}^{k} t_{n_i,p_i}}{t_{n,1}}$$

$$= \frac{[x^{n-|\Delta|}]\left(\prod_{i=1}^{k} T_{p_i}(x)\right)}{[x^n]T_1(x)}$$

$$\to \left(\prod_{i=1}^{k} T_{p_i}(\rho)\right) \cdot \sum_{j=i}^{k} \frac{C_j\, \rho^{|\Delta|}}{C_1\, T_{p_j}(\rho)}$$

The Uniform Infinite Planar Triangulation

$$\mathbb{P}_n\Bigg(B_r(\mathbf{T}) =$$



$$\Bigg) = \frac{\displaystyle\sum_{n_1+\cdots+n_k=n-|\Delta|} \prod_{i=1}^{k} t_{n_i,p_i}}{t_{n,1}}$$

$$= \frac{[x^{n-|\Delta|}]\left(\prod_{i=1}^{k} T_{p_i}(x)\right)}{[x^n]T_1(x)}$$

$$\to \left(\prod_{i=1}^{k} T_{p_i}(\rho)\right) \cdot \sum_{j=i}^{k} \frac{C_j\,\rho^{|\Delta|}}{C_1\,T_{p_j}(\rho)}$$

[Angel Schramm 03]

- First proof of weak convergence
- One end a.s.
- Spatial Markov Property: "the part of the UIPT inside a simple cycle is independent of the rest of the triangulation and its law only depends on the length of the cycle"

Some properties of the UIPT (and related models)

Metric properties:

- Volume (nb. of vertices) and perimeters of balls known to some extent.

  For example $\mathbb{E}\left[|B_r(\mathbf{T}_\infty)|\right] \sim \frac{2}{7} r^4$ [Curien – Le Gall 12]

Some properties of the UIPT (and related models)

Metric properties:

- Volume (nb. of vertices) and perimeters of balls known to some extent.

  For example $\mathbb{E}\left[|B_r(\mathbf{T}_\infty)|\right] \sim \frac{2}{7}r^4$ [Curien – Le Gall 12]

- Volume of hulls explicit [M. 16]
- "Uniqueness" of geodesic rays and horofunctions [Curien – M. 18]

Some properties of the UIPT (and related models)

Metric properties:

- Volume (nb. of vertices) and perimeters of balls known to some extent.

  For example $\mathbb{E}\left[|B_r(\mathbf{T}_\infty)|\right] \sim \dfrac{2}{7} r^4$ [Curien – Le Gall 12]

- Volume of hulls explicit [M. 16]
- "Uniqueness" of geodesic rays and horofunctions [Curien – M. 18]

Bond and site percolation well understood
[Angel, Angel–Curien, M.–Nolin, ...]

For example $p_c^{\text{site}} = 1/2$ and $p_c^{\text{bond}} = (2\sqrt{3} - 1)/11$ [Bernardi – Curien – Miermont 17]

Some properties of the UIPT (and related models)

Metric properties:

- Volume (nb. of vertices) and perimeters of balls known to some extent.

  For example $\mathbb{E}\left[|B_r(\mathbf{T}_\infty)|\right] \sim \frac{2}{7} r^4$ [Curien – Le Gall 12]

- Volume of hulls explicit [M. 16]
- "Uniqueness" of geodesic rays and horofunctions [Curien – M. 18]

Bond and site percolation well understood
[Angel, Angel–Curien, M.–Nolin, ...]

For example $p_c^{\text{site}} = 1/2$ and $p_c^{\text{bond}} = (2\sqrt{3} - 1)/11$ [Bernardi – Curien – Miermont 17]

Simple random Walk is recurrent [Gurel - Gurevich and Nachmias 13]

Adding matter: Ising model

Adding matter: Ising model on triangulations

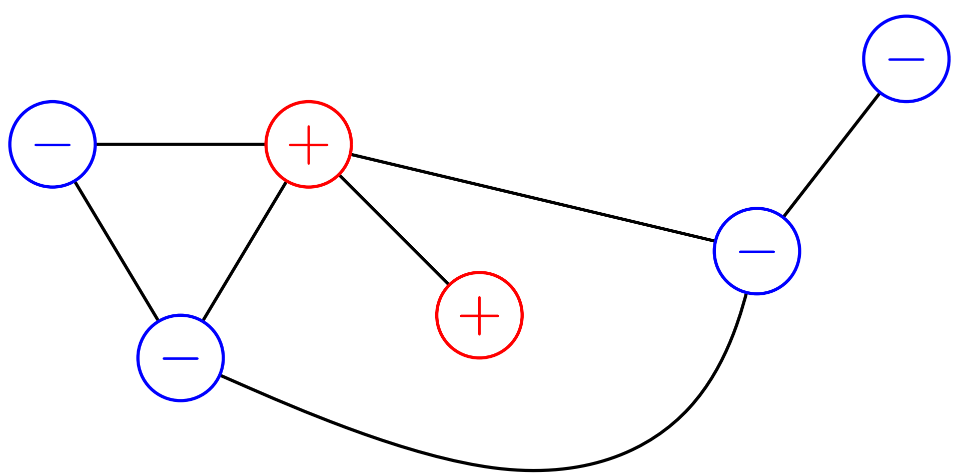
How does Ising model influence the underlying map?

Adding matter: Ising model on triangulations

How does Ising model influence the underlying map?

First, Ising model on a finite deterministic graph:

$G = (V, E)$ finite graph



Spin configuration on G:

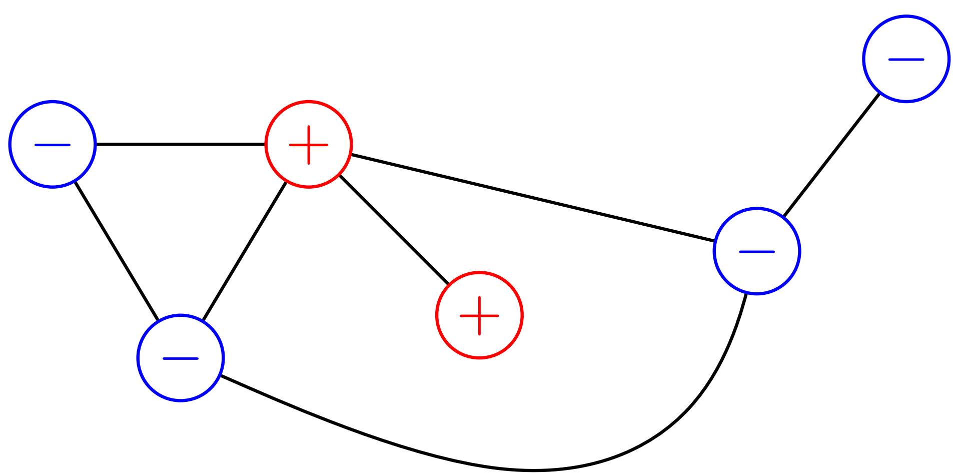
$$\sigma : V \to \{-1, +1\}.$$

Adding matter: Ising model on triangulations

How does Ising model influence the underlying map?

First, Ising model on a finite deterministic graph:

$G = (V, E)$ finite graph



Spin configuration on G:

$$\sigma : V \to \{-1, +1\}.$$

Ising model on G: take a random spin configuration with probability

$$P(\sigma) \propto e^{-\frac{\beta}{2} \sum_{v \sim v'} \mathbf{1}_{\{\sigma(v) \neq \sigma(v')\}}}$$

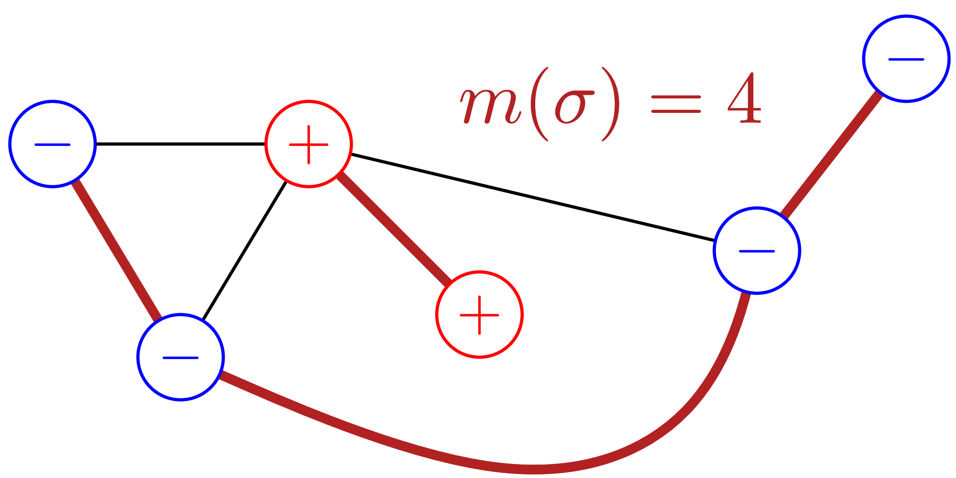
$\beta > 0$: inverse temperature.

Adding matter: Ising model on triangulations

How does Ising model influence the underlying map?

First, Ising model on a finite deterministic graph:

$G = (V, E)$ finite graph



Spin configuration on G:

$$\sigma : V \to \{-1, +1\}.$$

Ising model on G: take a random spin configuration with probability

$$P(\sigma) \propto e^{-\frac{\beta}{2} \sum_{v \sim v'} \mathbf{1}_{\{\sigma(v) \neq \sigma(v')\}}}$$

$\beta > 0$: inverse temperature.

Combinatorial formulation: $P(\sigma) \propto \nu^{m(\sigma)}$
with $m(\sigma) =$ number of monochromatic edges and $\nu = e^{\beta}$.

Adding matter: Ising model on triangulations

$\mathcal{T}_n = \{$rooted planar triangulations with $3n$ **edges**$\}$.

Random triangulation in $\mathcal{T}_n$ with probability $\propto \nu^{m(T,\sigma)}$?

Adding matter: Ising model on triangulations

$\mathcal{T}_n = \{$rooted planar triangulations with $3n$ **edges**$\}$.

Random triangulation in $\mathcal{T}_n$ with probability $\propto \nu^{m(T,\sigma)}$?

Generating series of **Ising-weighted triangulations**:

$$Q(\nu, t) = \sum_{T \in \mathcal{T}_f} \sum_{\sigma : V(T) \to \{-1,+1\}} \nu^{m(T,\sigma)} t^{e(T)}.$$

Adding matter: Ising model on triangulations

$\mathcal{T}_n = \{$rooted planar triangulations with $3n$ **edges**$\}$.

Random triangulation in $\mathcal{T}_n$ with probability $\propto \nu^{m(T,\sigma)}$?

Generating series of **Ising-weighted triangulations**:

$$Q(\nu,t) = \sum_{T\in\mathcal{T}_f} \sum_{\sigma:V(T)\to\{-1,+1\}} \nu^{m(T,\sigma)} t^{e(T)}.$$

Theorem [Bernardi – Bousquet-Mélou 11]
For every ν the series $Q(\nu,t)$ is algebraic, has $\rho_\nu > 0$ as unique dominant singularity and satisfies

$$[t^{3n}]Q(\nu,t) \underset{n\to\infty}{\sim} \begin{cases} \kappa\,\rho_{\nu_c}^{-n}\, n^{-7/3} & \text{if } \nu = \nu_c := 1+\frac{1}{\sqrt{7}}, \\ \kappa\,\rho_{\nu}^{-n}\, n^{-5/2} & \text{if } \nu \neq \nu_c. \end{cases}$$

This suggests an unusual behavior of the underlying maps for $\nu = \nu_c$.
See also [Boulatov – Kazakov 1987], [Bousquet-Mélou – Schaeffer 03] and [Bouttier – Di Francesco – Guitter 04].

Adding matter: Watabiki's predictions

Counting exponent:

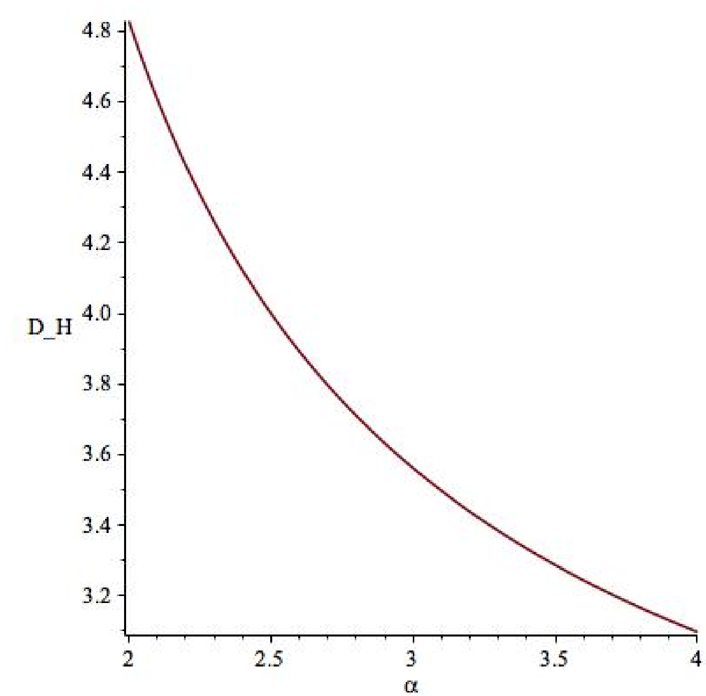
coeff $[t^n]$ of generating series of (decorated) maps $\sim \kappa\rho^{-n}n^{-\alpha}$

Central charge c:

$$\alpha = \frac{25 - c + \sqrt{(1-c)(25-c)}}{12}$$

Hausdorff dimension: [Watabiki 93]

$$D_H = 2\frac{\sqrt{25-c} + \sqrt{49-c}}{\sqrt{25-c} + \sqrt{1-c}}$$



Adding matter: Watabiki's predictions

Counting exponent:

coeff $[t^n]$ of generating series of (decorated) maps $\sim \kappa \rho^{-n} n^{-\alpha}$

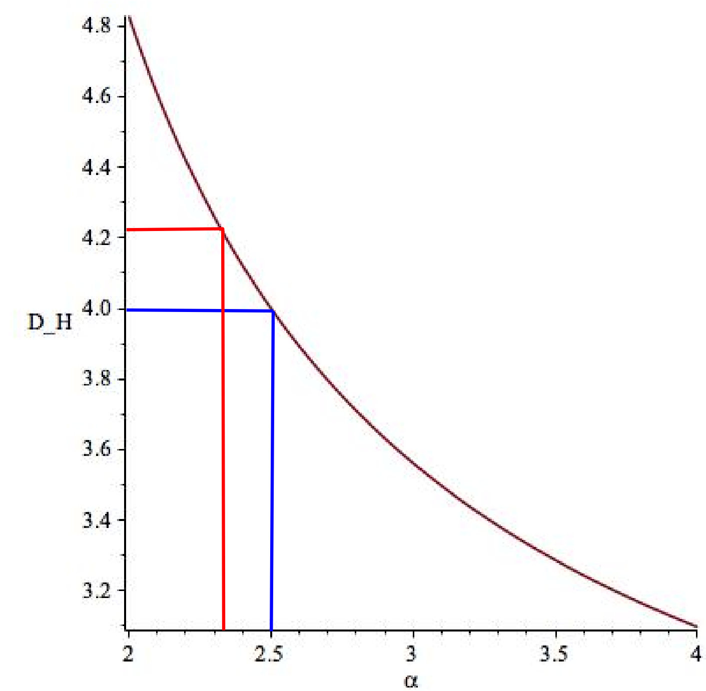
Central charge c:

$$\alpha = \frac{25 - c + \sqrt{(1-c)(25-c)}}{12}$$

Hausdorff dimension: [Watabiki 93]

$$D_H = 2\frac{\sqrt{25-c}+\sqrt{49-c}}{\sqrt{25-c}+\sqrt{1-c}}$$

- $\alpha = 5/2$ gives $D_H = 4$
- $\alpha = 7/3$ gives $D_H = \frac{7+\sqrt{97}}{4} \approx 4.21$



Local convergence of triangulations with spins

Probability measure on triangulations of $\mathcal{T}_n$ with a spin configuration:

$$\mathbb{P}_n^{\nu}\Big(\{(T,\sigma)\}\Big) = \frac{\nu^{m(T,\sigma)}}{[t^{3n}]Q(\nu,t)}.$$

Local convergence of triangulations with spins

Probability measure on triangulations of $\mathcal{T}_n$ with a spin configuration:

$$\mathbb{P}_n^{\nu}\Big(\{(T,\sigma)\}\Big) = \frac{\nu^{m(T,\sigma)}}{[t^{3n}]Q(\nu,t)}.$$

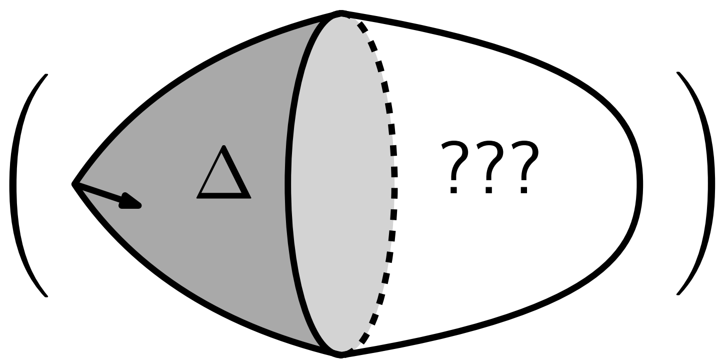
The plan:

- Local weak convergence of $\mathbb{P}_n^{\nu}$.
- Will be one-ended ($\alpha > 2$).
- Can we verify Watabiki's prediction ?

Local convergence and generating series

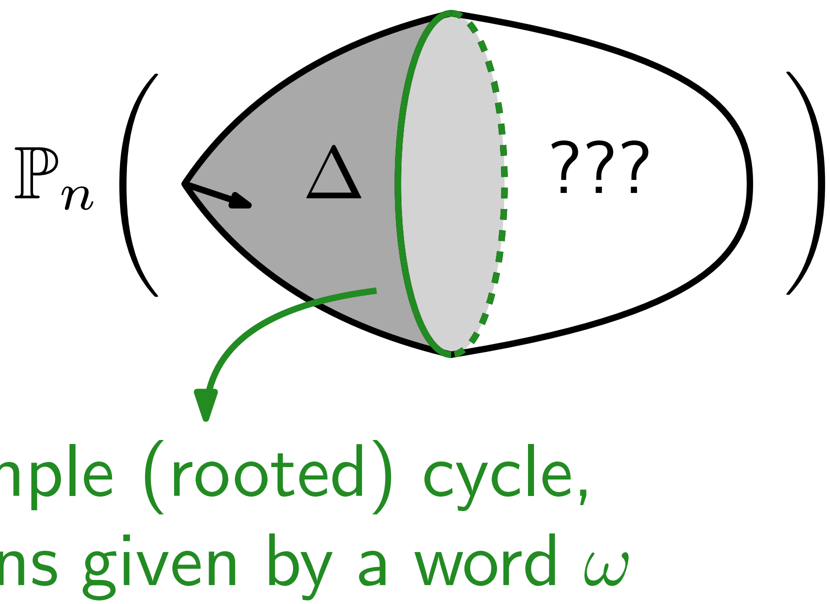
Need to evaluate, for every possible ball Δ
(here, one boundary to keep it simple)

$\mathbb{P}_n$



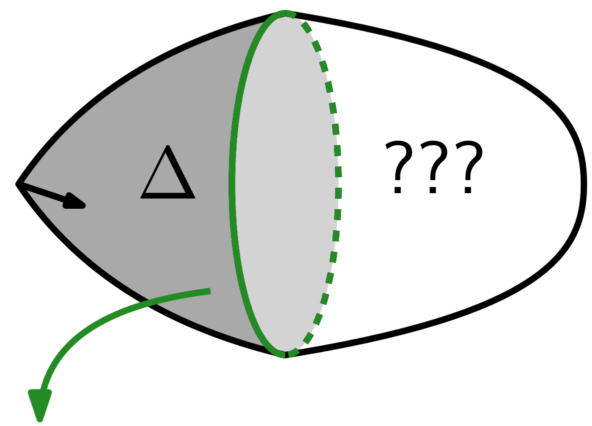
Local convergence and generating series

Need to evaluate, for every possible ball Δ
(here, one boundary to keep it simple)



Local convergence and generating series

Need to evaluate, for every possible ball Δ
(here, one boundary to keep it simple)

$\mathbb{P}_n\Big($ 

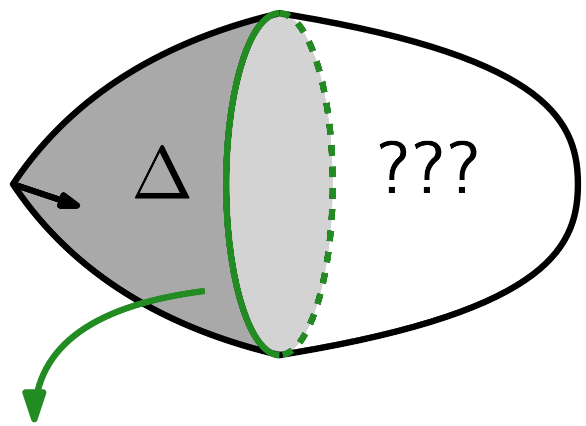
$\Big) = \dfrac{\nu^{m(\Delta)-m(\omega)}\,[t^{3n-e(\Delta)+|\omega|}]\mathbf{Z}_\omega(\nu,t)}{[t^{3n}]Q(\nu,t)}$

Simple (rooted) cycle,
spins given by a word ω

$\mathbf{Z}_\omega(\nu,t)$:= generating series of
triangulations with simple boundary ω

Local convergence and generating series

Need to evaluate, for every possible ball Δ
(here, one boundary to keep it simple)



$$\mathbb{P}_n\left(\cdots \right) = \frac{\nu^{m(\Delta)-m(\omega)}\,[t^{3n-e(\Delta)+|\omega|}]\mathbf{Z}_\omega(\nu,t)}{[t^{3n}]Q(\nu,t)}$$

Simple (rooted) cycle,
spins given by a word ω

$\mathbf{Z}_\omega(\nu,t) :=$ generating series of
triangulations with simple boundary ω

Theorem [Albenque – M. – Schaeffer 18+]
For every ω and ν, the series $t^{|\omega|}Z_\omega(\nu,t)$ is algebraic, has $\rho_\nu = t_\nu^3$ as unique dominant singularity and satisfies

$$[t^{3n}]t^{|\omega|}Z_\omega(\nu,t) \underset{n\to\infty}{\sim} \begin{cases} \kappa_\omega(\nu_c)\,\rho_{\nu_c}^{-n}\,n^{-7/3} & \text{if } \nu=\nu_c := 1+\frac{1}{\sqrt{7}}, \\ \kappa_\omega(\nu)\,\rho_\nu^{-n}\,n^{-5/2} & \text{if } \nu\neq\nu_c. \end{cases}$$

Triangulations with simple boundary

Fix a word ω, with injections from and into triangulations of the sphere:

$$[t^{3n}]t^{|\omega|}Z_\omega = \Theta\left(\rho_\nu^{-n} n^{-\alpha}\right), \text{ with } \alpha = 5/2 \text{ of } 7/3 \text{ depending on } \nu.$$

To get exact asymptotics we need, as series in t^3,

1. algebraicity,
2. no other dominant singularity than ρ_ν.

Triangulations with simple boundary

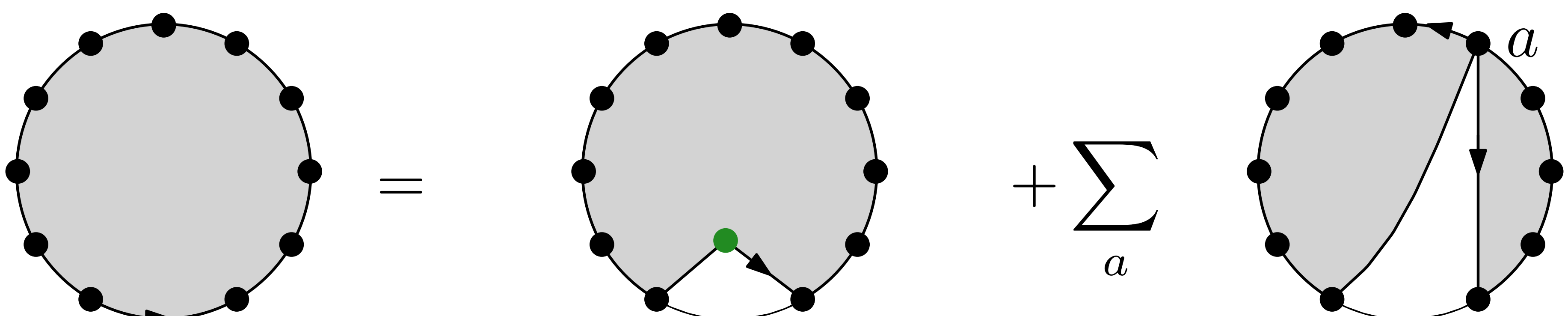
Fix a word ω, with injections from and into triangulations of the sphere:

$$[t^{3n}]t^{|\omega|}Z_\omega = \Theta\left(\rho_\nu^{-n} n^{-\alpha}\right), \text{ with } \alpha = 5/2 \text{ of } 7/3 \text{ depending on } \nu.$$

To get exact asymptotics we need, as series in t^3,

1. algebraicity,
2. no other dominant singularity than ρ_ν.

Tutte's equation (or peeling equation, or loop equation...):



$$Z_\omega = \left(Z_{\oplus\omega} + Z_{\ominus\omega} + \sum_{\omega=\omega_1 a \omega_2} Z_{a\omega_1} \cdot Z_{a\omega_2}\right) \times \nu^{\mathbf{1}_{\overleftarrow{\varpi}=\overrightarrow{\varpi}}}\, t$$

Triangulations with simple boundary

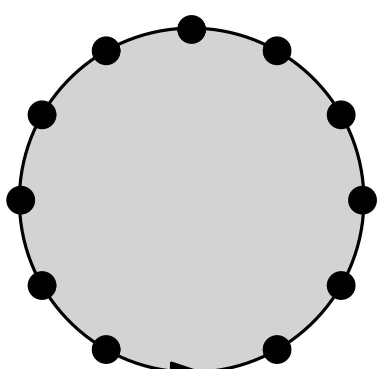
Fix a word ω, with injections from and into triangulations of the sphere:

$$[t^{3n}]t^{|\omega|}Z_\omega = \Theta\left(\rho_\nu^{-n} n^{-\alpha}\right), \text{ with } \alpha = 5/2 \text{ of } 7/3 \text{ depending on } \nu.$$

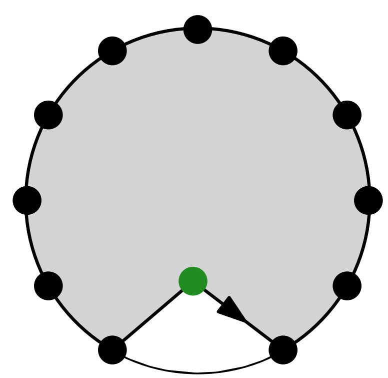
To get exact asymptotics we need, as series in t^3,

1. algebraicity,
2. no other dominant singularity than ρ_ν.

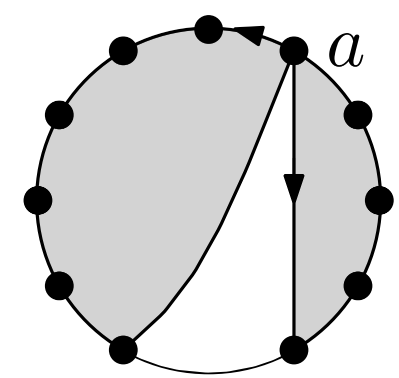
Tutte's equation (or peeling equation, or loop equation...):



$=$



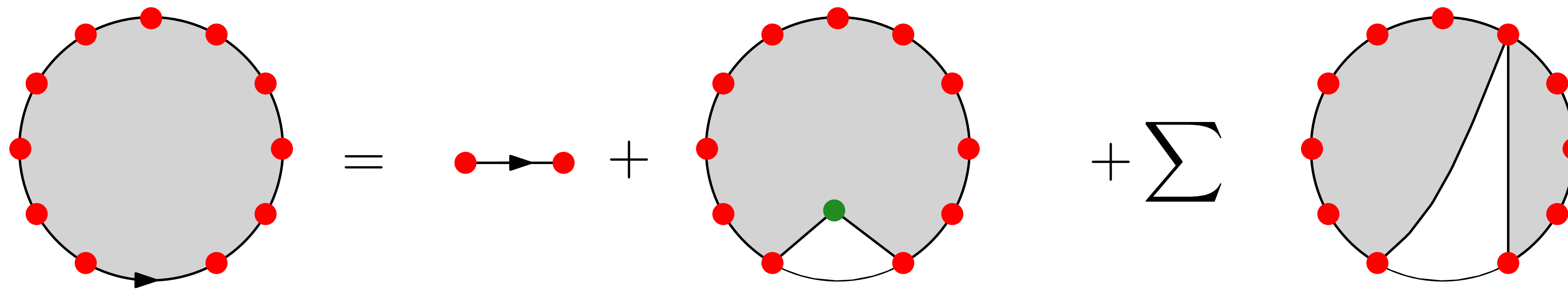
$+\sum_a$



$$Z_\omega = \left(Z_{\oplus\omega} + Z_{\ominus\omega} + \sum_{\omega=\omega_1 a \omega_2} Z_{a\omega_1} \cdot Z_{a\omega_2} \right) \times \nu^{\mathbf{1}_{\overleftarrow{\omega}=\overrightarrow{\omega}}}\, t$$

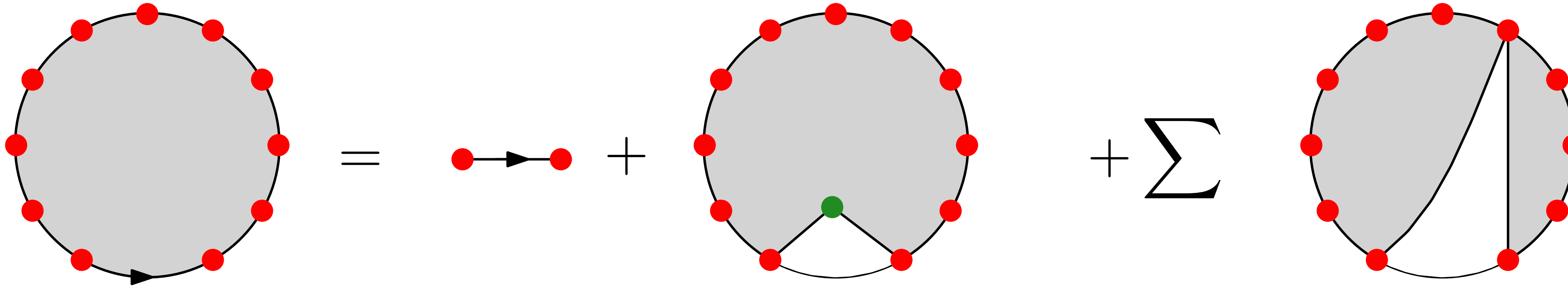
Double induction on $|\omega|$ and number of $\ominus$'s: enough to prove 1. and 2. for the $t^p Z_{\oplus^p}$'s

Positive boundary conditions: two catalytic variables



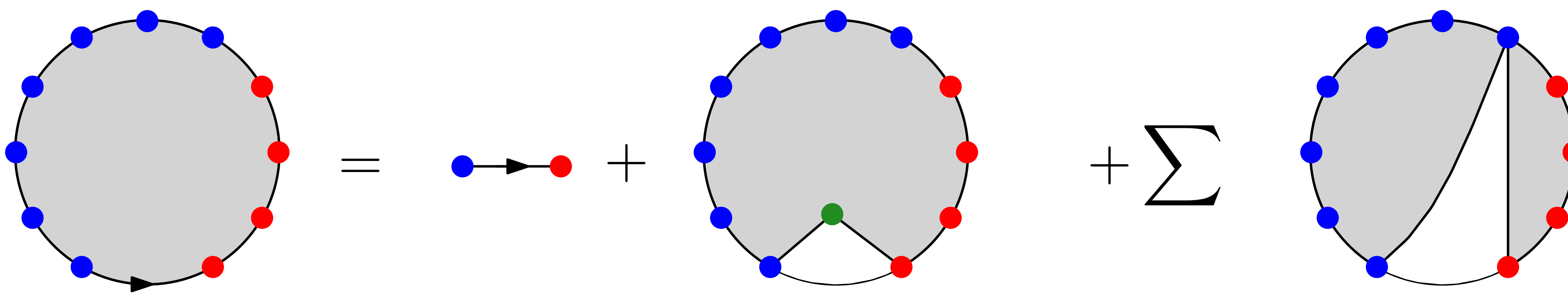
$$A(x) := \sum_{p \geq 1} Z_{\oplus^p} x^p = \nu t x^2 + \quad + \frac{\nu t}{x} (A(x))^2$$

Positive boundary conditions: two catalytic variables



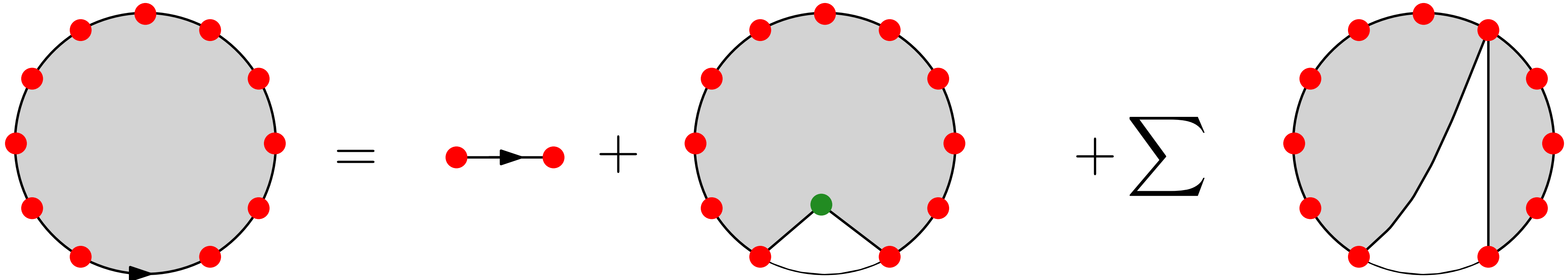
$$A(x) := \sum_{p\geq 1} Z_{\oplus^p} x^p = \nu t x^2 + \qquad + \frac{\nu t}{x}(A(x))^2$$

Peeling equation **at interface** $\ominus$–$\oplus$:



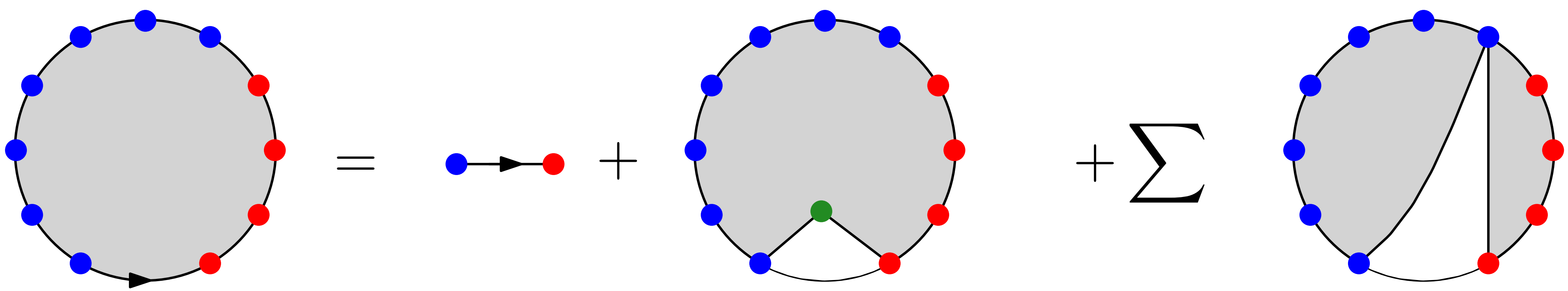
$$S(x,y) := \sum_{p,q\geq 1} Z_{\oplus^p \ominus^q} x^p y^q$$

Positive boundary conditions: two catalytic variables



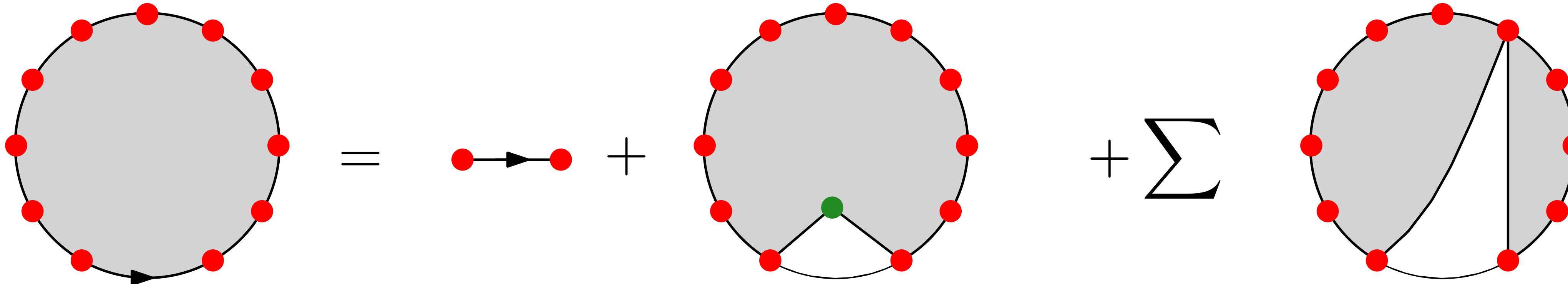
$$A(x) := \sum_{p\geq 1} Z_{\oplus^p} x^p = \nu t x^2 + \frac{\nu t}{x}\left(A(x) - x Z_{\oplus}\right) + \nu t\, [y] S(x,y) + \frac{\nu t}{x}(A(x))^2$$

Peeling equation **at interface** $\ominus$–$\oplus$:



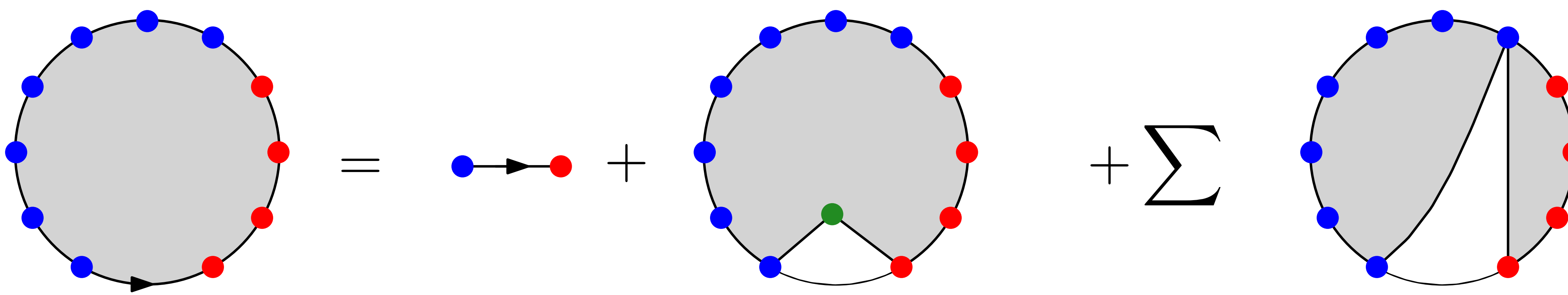
$$S(x,y) := \sum_{p,q\geq 1} Z_{\oplus^p \ominus^q} x^p y^q$$

Positive boundary conditions: two catalytic variables



$$A(x) := \sum_{p\geq 1} Z_{\oplus^p} x^p = \nu t x^2 + \frac{\nu t}{x}\left(A(x) - x Z_{\oplus}\right) + \nu t\,[y]S(x,y) + \frac{\nu t}{x}(A(x))^2$$

Peeling equation **at interface** $\ominus$–$\oplus$:



$$S(x,y) := \sum_{p,q\geq 1} Z_{\oplus^p \ominus^q} x^p y^q$$

$$= txy + \frac{t}{x}\Big(S(x,y) - x[x]S(x,y)\Big) + \frac{t}{y}\Big(S(x,y) - y[y]S(x,y)\Big) + \frac{t}{x}S(x,y)A(x) + \frac{t}{y}S(x,y)A(y)$$

From two catalytic variables to one: Tutte's invariants

Kernel method: equation for S reads

$$K(x, y) \cdot S(x, y) = R(x, y)$$

where $$K(x, y) = 1 - \frac{t}{x} - \frac{t}{y} - \frac{t}{x}A(x) - \frac{t}{y}A(y).$$

From two catalytic variables to one: Tutte's invariants

Kernel method: equation for S reads

$$K(x,y)\cdot S(x,y) = R(x,y)$$

where $$K(x,y) = 1 - \frac{t}{x} - \frac{t}{y} - \frac{t}{x}A(x) - \frac{t}{y}A(y).$$

1. Find **two** series Y_1 and Y_2 in $\mathbb{Q}(x)[[t]]$ such that $K(x, Y_i/t) = 0$.

From two catalytic variables to one: Tutte's invariants

Kernel method: equation for S reads

$$K(x,y)\cdot S(x,y)=R(x,y)$$

where $$K(x,y)=1-\frac{t}{x}-\frac{t}{y}-\frac{t}{x}A(x)-\frac{t}{y}A(y).$$

1. Find **two** series Y_1 and Y_2 in $\mathbb{Q}(x)[[t]]$ such that $K(x,Y_i/t)=0$.

   It gives $\frac{1}{Y_1}\left(A(Y_1/t)+1\right)=\frac{1}{Y_2}\left(A(Y_2/t)+1\right)$.

From two catalytic variables to one: Tutte's invariants

Kernel method: equation for S reads

$$K(x,y)\cdot S(x,y) = R(x,y)$$

where $$K(x,y) = 1 - \frac{t}{x} - \frac{t}{y} - \frac{t}{x}A(x) - \frac{t}{y}A(y).$$

1. Find **two** series Y_1 and Y_2 in $\mathbb{Q}(x)[[t]]$ such that $K(x, Y_i/t) = 0$.

   It gives $\frac{1}{Y_1}\left(A(Y_1/t)+1\right) = \frac{1}{Y_2}\left(A(Y_2/t)+1\right)$.

   $I(y) := \frac{1}{y}\left(A(y/t)+1\right)$ is called an **invariant**.

From two catalytic variables to one: Tutte's invariants

Kernel method: equation for S reads

$$K(x,y)\cdot S(x,y) = R(x,y)$$

where $$K(x,y) = 1 - \frac{t}{x} - \frac{t}{y} - \frac{t}{x}A(x) - \frac{t}{y}A(y).$$

1. Find **two** series Y_1 and Y_2 in $\mathbb{Q}(x)[[t]]$ such that $K(x, Y_i/t) = 0$.

   It gives $\frac{1}{Y_1}\left(A(Y_1/t)+1\right) = \frac{1}{Y_2}\left(A(Y_2/t)+1\right)$.

   $I(y) := \frac{1}{y}\left(A(y/t)+1\right)$ is called an **invariant**.

2. Work a bit with the help of $R(x, Y_i/t) = 0$ to get a **second invariant** $J(y)$ depending only on $t, Z_{\oplus}(t), y$ and $A(y/t)$.

From two catalytic variables to one: Tutte's invariants

Kernel method: equation for S reads

$$K(x,y)\cdot S(x,y)=R(x,y)$$

where $$K(x,y)=1-\frac{t}{x}-\frac{t}{y}-\frac{t}{x}A(x)-\frac{t}{y}A(y).$$

1. Find **two** series Y_1 and Y_2 in $\mathbb{Q}(x)[[t]]$ such that $K(x,Y_i/t)=0$.

   It gives $\frac{1}{Y_1}\left(A(Y_1/t)+1\right)=\frac{1}{Y_2}\left(A(Y_2/t)+1\right)$.

   $I(y):=\frac{1}{y}\left(A(y/t)+1\right)$ is called an **invariant**.

2. Work a bit with the help of $R(x,Y_i/t)=0$ to get a **second invariant** $J(y)$ depending only on $t, Z_{\oplus}(t), y$ and $A(y/t)$.

3. Prove that $J(y)=C_0(t)+C_1(t)I(y)+C_2(t)I^2(y)$ with C_i's explicit polynomials in $t, Z_{\oplus}(t)$ and $Z_{\oplus^2}(t)$.

Equation with one catalytic variable for $A(y)$ with $Z_{\oplus}$ and $Z_{\oplus^2}$!

Explicit solution for positive boundary conditions

Equation with one catalytic variable reads:

$$2t^2\nu(1-\nu)\left(\frac{A(y)}{y} - Z_\oplus\right) = y \cdot \mathrm{Pol}\left(\nu, \frac{A(y)}{y}, Z_\oplus, Z_{\oplus^2}, t, y\right)$$

[Bousquet-Mélou – Jehanne 06] gives **algebraicity** and strategy to solve this kind of equation.

Explicit solution for positive boundary conditions

Equation with one catalytic variable reads:

$$2t^2\nu(1-\nu)\left(\frac{A(y)}{y} - Z_{\oplus}\right) = y \cdot \mathrm{Pol}\left(\nu, \frac{A(y)}{y}, Z_{\oplus}, Z_{\oplus^2}, t, y\right)$$

[Bousquet-Mélou – Jehanne 06] gives **algebraicity** and strategy to solve this kind of equation.

Much easier: [Bernardi – Bousquet Mélou 11] gives us $Z_{\oplus}$ and $Z_{\oplus^2}$!

Explicit solution for positive boundary conditions

Equation with one catalytic variable reads:

$$2t^2\nu(1-\nu)\left(\frac{A(y)}{y} - Z_{\oplus}\right) = y \cdot \mathrm{Pol}\left(\nu, \frac{A(y)}{y}, Z_{\oplus}, Z_{\oplus^2}, t, y\right)$$

[Bousquet-Mélou – Jehanne 06] gives **algebraicity** and strategy to solve this kind of equation.

Much easier: [Bernardi – Bousquet Mélou 11] gives us $Z_{\oplus}$ and $Z_{\oplus^2}$!

Maple: **rational (and Lagrangian) parametrization !**

$$t^3 = U\frac{P_1(\mu, U)}{4(1-2U)^2(1+\mu)^3}$$

$$y = V\frac{P_2(\mu, U, V)}{(1-2U)(1+\mu)^2(1-V)^2}$$

$$t^3 A(t, ty) = \frac{V P_3(\mu, U, V)}{4(1-2U)^2(1+\mu)^3(1-V)^3}$$

with $\nu = \frac{1+\mu}{1-\mu}$ and P_i's explicit polynomials.

Going back to local convergence

1. Fix $r \geq 0$ and take Δ a r-ball with boundary spins $\partial\Delta = (\omega_1, \ldots, \omega_k)$:

$$\mathbb{P}_n\left(B_r(T,v) = \Delta\right) = \frac{\nu^{m(\Delta)-m(\partial\Delta)}\,[t^{3n-e(\Delta)+|\partial\Delta|}]\left(\prod_{i=1}^{k} Z_{\omega_i}(\nu,t)\right)}{[t^{3n}]Q(\nu,t)}$$

$$\underset{n\to\infty}{\to} \left(\prod_{i=1}^{k} Z_{\omega_i}(\nu,t_\nu)\right) \cdot \sum_{j=1}^{k} \frac{\nu^{m(\Delta)-m(\partial\Delta)}\, t_\nu^{|\Delta|-|\omega|}\ \kappa_{\omega_j}}{\kappa\, t_\nu^{|\omega_j|} Z_{\omega_j}(\nu,t_\nu)}.$$

Going back to local convergence

1. Fix $r \geq 0$ and take Δ a r-ball with boundary spins $\partial\Delta = (\omega_1, \ldots, \omega_k)$:

$$\mathbb{P}_n\left(B_r(T,v) = \Delta\right) = \frac{\nu^{m(\Delta)-m(\partial\Delta)}\,[t^{3n-e(\Delta)+|\partial\Delta|}]\left(\prod_{i=1}^{k} Z_{\omega_i}(\nu,t)\right)}{[t^{3n}]Q(\nu,t)}$$

$$\underset{n\to\infty}{\to} \left(\prod_{i=1}^{k} Z_{\omega_i}(\nu,t_\nu)\right) \cdot \sum_{j=1}^{k} \frac{\nu^{m(\Delta)-m(\partial\Delta)}\,t_\nu^{|\Delta|-|\omega|}\,\kappa_{\omega_j}}{\kappa\,t_\nu^{|\omega_j|} Z_{\omega_j}(\nu,t_\nu)}.$$

2. Remains to prove tightness.

Going back to local convergence

1. Fix $r \geq 0$ and take Δ a r-ball with boundary spins $\partial\Delta = (\omega_1, \ldots, \omega_k)$:

$$\mathbb{P}_n\left(B_r(T,v) = \Delta\right) = \frac{\nu^{m(\Delta)-m(\partial\Delta)}\left[t^{3n-e(\Delta)+|\partial\Delta|}\right]\left(\prod_{i=1}^k Z_{\omega_i}(\nu,t)\right)}{[t^{3n}]Q(\nu,t)}$$

$$\underset{n\to\infty}{\rightarrow} \left(\prod_{i=1}^k Z_{\omega_i}(\nu,t_\nu)\right) \cdot \sum_{j=1}^k \frac{\nu^{m(\Delta)-m(\partial\Delta)}\, t_\nu^{|\Delta|-|\omega|}\, \kappa_{\omega_j}}{\kappa\, t_\nu^{|\omega_j|} Z_{\omega_j}(\nu,t_\nu)}.$$

2. Remains to prove tightness.

   - Maps are uniformly rooted:
     tightness of root degree is enough

Going back to local convergence

1. Fix $r \geq 0$ and take Δ a r-ball with boundary spins $\partial\Delta = (\omega_1, \ldots, \omega_k)$:

$$\mathbb{P}_n\left(B_r(T,v) = \Delta\right) = \frac{\nu^{m(\Delta)-m(\partial\Delta)}\,[t^{3n-e(\Delta)+|\partial\Delta|}]\left(\prod_{i=1}^k Z_{\omega_i}(\nu,t)\right)}{[t^{3n}]Q(\nu,t)}$$

$$\underset{n\to\infty}{\to}\left(\prod_{i=1}^k Z_{\omega_i}(\nu,t_\nu)\right)\cdot\sum_{j=1}^k \frac{\nu^{m(\Delta)-m(\partial\Delta)}\,t_\nu^{|\Delta|-|\omega|}\quad\kappa_{\omega_j}}{\kappa\,t_\nu^{|\omega_j|}Z_{\omega_j}(\nu,t_\nu)}.$$

2. Remains to prove tightness.
   - Maps are uniformly rooted: tightness of root degree is enough
   - We show that expected degree at the root under $\mathbb{P}_n$ is bounded with n

The story so far

What we know:

- Convergence in law for the local toplogy.
- The limiting random triangulation has one end *a.s.*

The story so far

What we know:

- Convergence in law for the local toplogy.
- The limiting random triangulation has one end *a.s.*
- A spatial Markov property.
- Some links with Boltzmann triangulations.

The story so far

What we know:

- Convergence in law for the local toplogy.
- The limiting random triangulation has one end *a.s.*
- A spatial Markov property.
- Some links with Boltzmann triangulations.

In progress:

- Recurrence of SRW (vertex degrees have exponential tails)
- Cluster properties.

The story so far

What we know:

- Convergence in law for the local toplogy.
- The limiting random triangulation has one end *a.s.*
- A spatial Markov property.
- Some links with Boltzmann triangulations.

In progress:

- Recurrence of SRW (vertex degrees have exponential tails)
- Cluster properties.

What we would like to know:

- Singularity with respect to the UIPT?
- Volume growth?

The story so far

What we know:

- Convergence in law for the local toplogy.
- The limiting random triangulation has one end *a.s.*
- A spatial Markov property.
- Some links with Boltzmann triangulations.

In progress:

- Recurrence of SRW (vertex degrees have exponential tails)
- Cluster properties.

What we would like to know:

- Singularity with respect to the UIPT?
- Volume growth?
- At least volume growth $\neq 4$ at ν_c?

Summer school **Random trees and graphs**
July 1 to 5, 2019 in Marseille France
Org. M. Albenque, J. Bettinelli, J. Rué and L.M.

Summer school **Random walks and models of complex networks**
July 8 to 19, 2019 in Nice
Org. B. Reed and D. Mitsche

Thank you for your attention!