Topological aspects of Colored Tensor Models

Paola Cristofori
University of Modena and Reggio Emilia

International Workshop on the
Physics and Mathematics of Discrete Geometries
Nagoya, November 5-9, 2018

joint works with M.R. Casali - S. Dartois - L. Grasselli

Outline

$(d+1)$-colored graphs

$(d+1)$-colored graph (Γ, γ)

- $\Gamma = (V(\Gamma), E(\Gamma))$ regular graph of degree $d+1$,
- $\gamma : E(\Gamma) \to \Delta_d = \{0, \ldots, d\}$ such that $\gamma(e) \neq \gamma(f)$ for each pair of adjacent edges $e, f \in E(\Gamma)$ (*edge-coloration*)

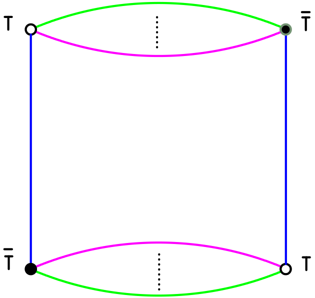
graph = multigraph (multiple edges allowed, loops forbidden)

Tensor invariants encoded by d-colored graphs

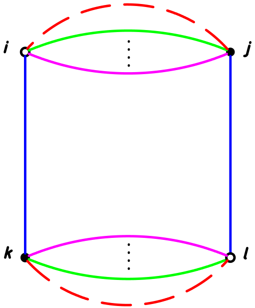
- White (black) vertices for T ($\overline{T}$)
- Edges colored by the position of the index

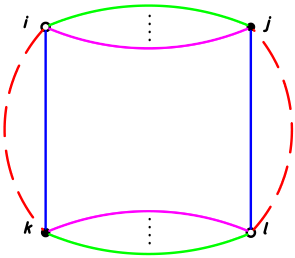
Example:

$$Q(T,\overline{T}) = \sum_{\substack{i_1,\dots,i_d=1\\ j_1,\dots,j_d=1}}^{N} \overline{T}_{i_1,i_2,\dots,i_d}\, T_{j_1,i_2,\dots,i_d}\, \overline{T}_{j_1,j_2,\dots,j_d}\, T_{i_1,j_2,\dots,j_d}$$



$$\langle Q(T,\overline{T})\rangle = \sum_{\{i_n,j_n,l_n,k_n=1\}_{1\leq n\leq d}}^{N} \delta_{i_1k_1}\delta_{j_1l_1}\left(\prod_{p=2}^{d}\delta_{i_pj_p}\right)\left(\prod_{q=2}^{d}\delta_{l_qk_q}\right)$$
$$\left(\langle \overline{T}_{i_1\dots i_d}\,T_{j_1\dots j_d}\rangle\langle \overline{T}_{l_1\dots l_d}\,T_{k_1\dots k_d}\rangle + \langle \overline{T}_{i_1\dots i_d}\,T_{k_1\dots k_d}\rangle\langle \overline{T}_{l_1\dots l_d}\,T_{j_1\dots j_d}\rangle\right)$$





Colored Tensor Models (CTM)

A $(d+1)$-dimensional Colored Tensor Model is a *formal partition function* defined by a formal integral:

$$\mathcal{Z}[N, \{t_B\}_{B \in \mathcal{CG}(d)}] := \int_f \frac{dT d\overline{T}}{(2\pi)^{N^d}} e^{-N^{d-1}\overline{T} \cdot T + \sum_B \alpha_B B(\overline{T}, T)}$$

where

- $T, \overline{T} \in \otimes_d \mathbb{C}^N$
- $\mathcal{CG}(d)$ is the set of bipartite d-colored graphs
- $B(\overline{T}, T)$ invariant encoded by a d-colored graph $B \in \mathcal{CG}(d)$

G-degree (R. Gurau)

If (Γ, γ) is a $(d+1)$-colored graph of order $2p$, set:

$$\omega_G(\Gamma) \;=\; \frac{(d-1)!}{2}\Big(d + \frac{d}{2}(d-1)p - \sum_{r,s\in\Delta_d} g_{rs}\Big)$$

g_{rs} = number of $\{r,s\}$-colored cycles of Γ

Theorem (Bonzom-Gurau)

If $\alpha_B = N^{d-1-\frac{2}{(d-2)!}\omega_G(B)} \frac{t_B}{|\mathrm{Aut}(B)|}$

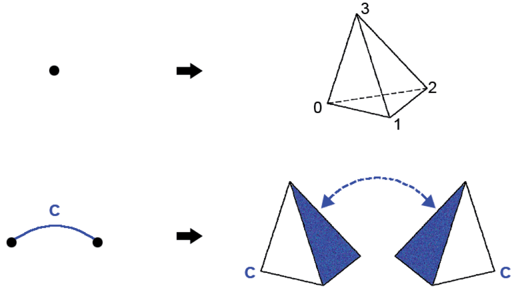
the *free energy* $\frac{1}{N^d} \log \mathcal{Z}[N, \{t_B\}]$ is

$$\sum_{\omega_G \geq 0} N^{-\frac{2}{(d-1)!}\omega_G} F_{\omega_G}[\{t_B\}] \in \mathbb{C}[[N^{-1}, \{t_B\}]] \qquad (\frac{1}{N} \textit{ expansion})$$

where the coefficients $F_{\omega_G}[\{t_B\}]$ are generating functions of *connected* $(d+1)$-colored graphs with fixed Gurau degree ω_G

The pseudocomplex $K(\Gamma)$

1) take a d-simplex $\sigma(x)$ for every vertex $x \in V(\Gamma)$, and label its vertices by Δ_d;
2) if $x, y \in V(\Gamma)$ are joined by a c-colored edge, identify the $(d-1)$-faces of $\sigma(x)$ and $\sigma(y)$ opposite to c-labelled vertices, so that equally labelled vertices coincide.



- ⋆ $K(\Gamma)$ is a *(closed) d-pseudomanifold*
- ⋆ (Γ, γ) *represents* $|K(\Gamma)|$
- ⋆ If $M^d = |K(\Gamma)|$ is a closed manifold, (Γ, γ) is called a *gem* = "graph encoded manifold" of M^d.
- ⋆ Γ is the 1-skeleton of the dual complex of $K(\Gamma)$

- ⋆ $K(\Gamma)$ is a *(closed) d-pseudomanifold*
- ⋆ (Γ, γ) *represents* $|K(\Gamma)|$
- ⋆ If $M^d = |K(\Gamma)|$ is a closed manifold, (Γ, γ) is called a *gem* = "graph encoded manifold" of M^d.
- ⋆ Γ is the 1-skeleton of the dual complex of $K(\Gamma)$

Definition

A singular (PL) d-manifold ($d > 1$) is a compact connected d-dimensional polyhedron admitting a simplicial triangulation where the links of vertices are closed connected $(d-1)$-manifolds, while the links of all h-simplices with $h > 0$ are PL $(d-h-1)$-spheres.
Vertices whose links are not PL $(d-1)$-spheres are called singular.

Singular manifolds/bounded manifolds

{(closed) manifolds} $\subset$ {singular manifolds} $\subset$ {pseudomanifolds}

From a singular d-manifold N to a d-manifold with boundary $\check{N}$
(by deleting regular neighbourhoods of singular vertices)

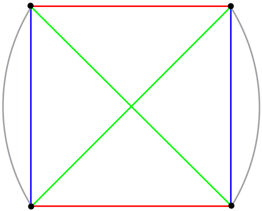
From a d-manifold M with boundary to a singular d-manifold $\hat{M}$
(by capping off the boundary components with cones over them)

PROPERTIES

- $K(\Gamma)$ is orientable iff Γ is bipartite
- for each $c \in \Delta_d$, the c-labelled vertices of $K(\Gamma)$ are in bijection with the connected components of $\Gamma_{\hat{c}} = \Gamma_{\Delta_d - \{c\}}$. ($\hat{c}$-residues of Γ).
  For each $\hat{c}$-residue Ξ, $K(\Xi)$ is PL isomorphic to $Lk(v_c, K'(\Gamma))$.
- $|K(\Gamma)|$ is a d-manifold (rep. singular d-manifold) if and only if, for every $c \in \Delta_d$, each connected component of $\Gamma_{\hat{c}}$ represents $\mathbb{S}^{d-1}$ (resp. is a gem of a closed $(d-1)$-manifold).

PROPERTIES

- $K(\Gamma)$ is orientable iff Γ is bipartite
- for each $c \in \Delta_d$, the c-labelled vertices of $K(\Gamma)$ are in bijection with the connected components of $\Gamma_{\hat{c}} = \Gamma_{\Delta_d - \{c\}}$. ($\hat{c}$-residues of Γ). For each $\hat{c}$-residue Ξ, $K(\Xi)$ is PL isomorphic to $Lk(v_c, K'(\Gamma))$.
- $|K(\Gamma)|$ is a d-manifold (rep. singular d-manifold) if and only if, for every $c \in \Delta_d$, each connected component of $\Gamma_{\hat{c}}$ represents $\mathbb{S}^{d-1}$ (resp. is a gem of a closed $(d-1)$-manifold).



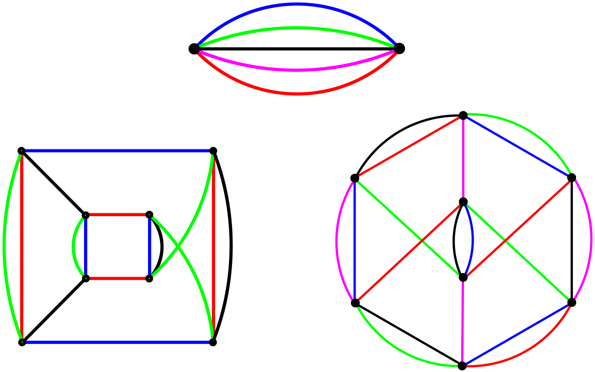
$\Sigma(\mathbb{RP}^2) =$ the suspension over $\mathbb{RP}^2$

Existence Theorem (Pezzana, Casali - C. - Grasselli)

Any orientable (resp. non-orientable) singular manifold N^d admits a bipartite (resp. non-bipartite) $(d+1)$-colored graph (Γ, γ) representing it. Moreover, (Γ, γ) can always be supposed to be contracted, i.e. for each color $c \in \Delta_d$, either $K(\Gamma)$ has only one c-colored vertex or all c-colored vertices of $K(\Gamma)$ are singular.

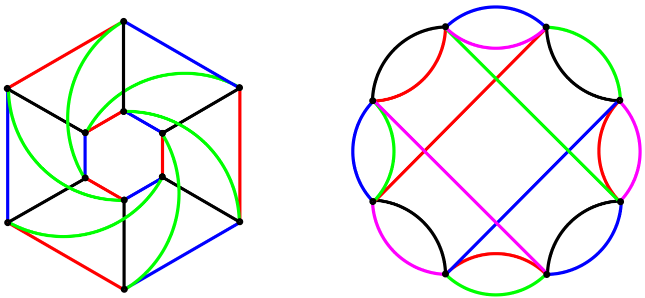
In the case of a (closed) manifold the pseudocomplex has exactly $d+1$ vertices. The corresponding graph is called a crystallization of $|K(\Gamma)|$.

Examples



Contracted gem of $\mathbb{S}^4, \mathbb{S}^2 \widetilde{\times} \mathbb{S}^1$ and $\mathbb{CP}^2$

Examples

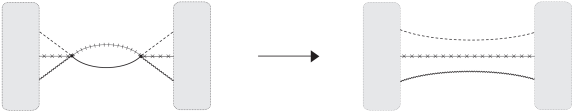


Contracted graphs representing $L(3,1)$ (left) and $\mathbb{S}^2 \times \mathbb{D}^2$ (right)

Dipole moves

h-dipole $(1 \leq h \leq d)$ of Γ

$\Theta = (V(\Theta) = \{v, w\}, E(\Theta) = \{e_{j_1}, \dots, e_{j_h}\}) \quad j_1, \dots, j_h \in \Delta_d$, such that v and w belong to different connected components of $\Gamma_{\Delta_d - \{j_1, \dots, j_h\}}$.



$\Gamma \xrightarrow{\textit{cancellation}} \Gamma' \qquad \Gamma \xleftarrow{\textit{insertion}} \Gamma'$

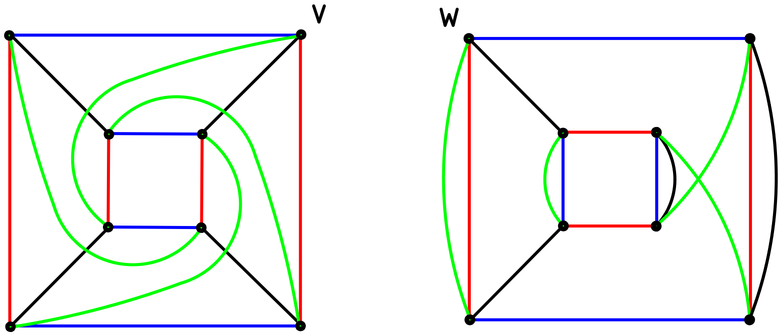
An h-dipole Θ is called *proper* if and only if $|K(\Gamma)|$ and $|K(\Gamma')|$ are PL-isomorphic.

Proper dipoles

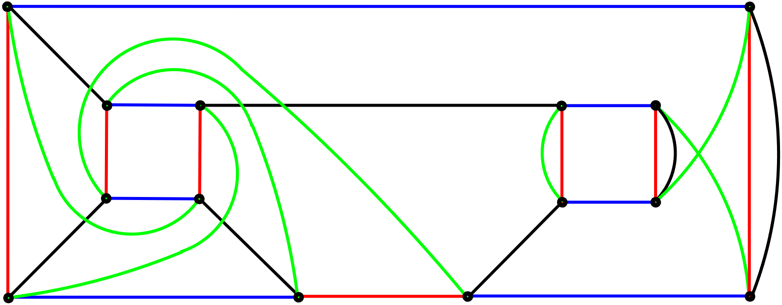
Gagliardi, 1987

Let Θ be an h-dipole of an $(d+1)$-colored graph (Γ, γ). If at least one of the connected components of $\Gamma_{\Delta_d - \{j_1, \dots, j_h\}}$ containing v or w represents a PL $(d-h)$-sphere then Θ is proper. As a consequence, any dipole of a gem of a closed PL manifold is proper.

Graph connected sum: $\mathbb{RP}^3 \# \mathbb{S}^2 \tilde{\times} \mathbb{S}^1$

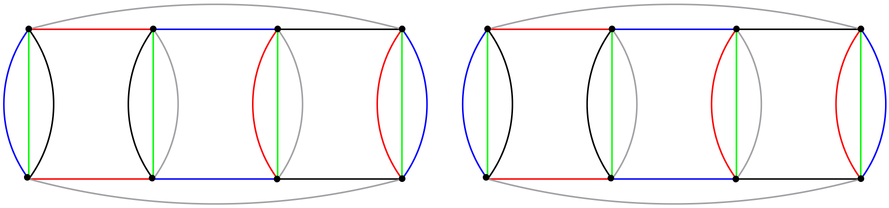


Graph connected sum: $\mathbb{RP}^3 \# \mathbb{S}^2 \tilde{\times} \mathbb{S}^1$



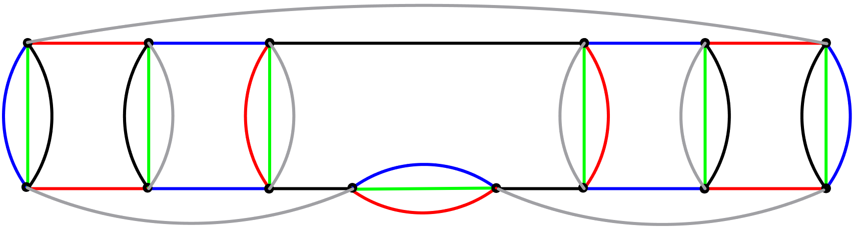
$\mathbb{Y}^4_1 = \mathbb{S}^1 \times \mathbb{D}^3$

Genus one 4-dimensional orientable handlebodies



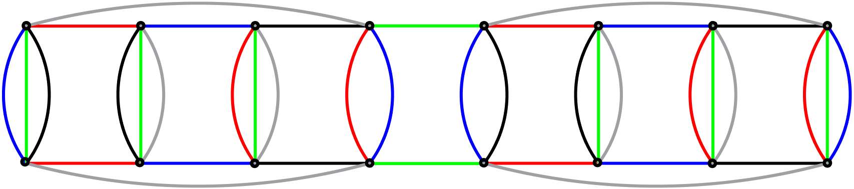
Connected sums I

$\mathbb{Y}_2^4$ as boundary connected sum



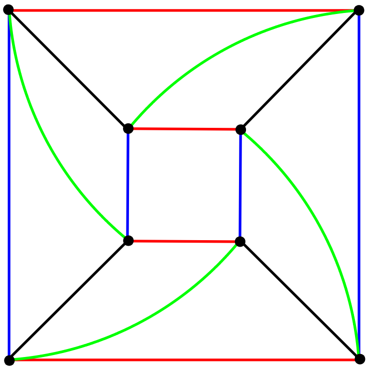
Connected sums II

The connected sum $\mathbb{Y}^4_1 \# \mathbb{Y}^4_1$



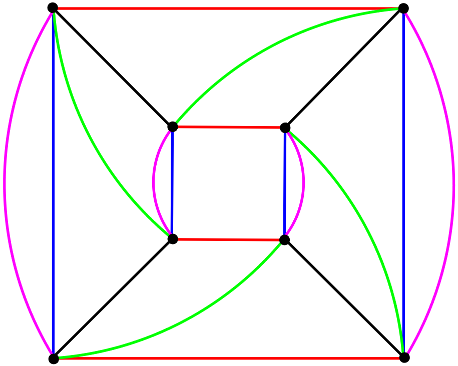
Suspension and product: $N^d = \Sigma(M^{d-1})$, $\check{N}^d = M^{d-1} \times I$

$$M^3 = \mathbb{RP}^3$$



Suspension and product: $N^d = \Sigma(M^{d-1})$, $\check{N}^d = M^{d-1} \times I$

$$\check{N}^4 = \mathbb{RP}^3 \times I$$



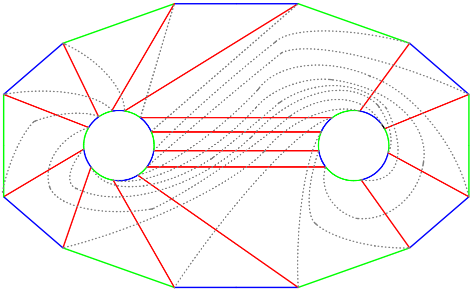
Regular embeddings (jackets)

A cellular embedding $\phi : |\Gamma| \to F$ of a $(d+1)$-colored graph (Γ, γ) into a (closed) surface F is called a regular embedding if there exists a cyclic permutation $\varepsilon = (\varepsilon_0, \dots, \varepsilon_d)$ of Δ_d s.t. each connected component of $F - \phi(|\Gamma|)$ is an open ball bounded by the image of an $\{\varepsilon_i, \varepsilon_{i+1}\}-$ colored cycle of Γ $(\forall i \in \mathbb{Z}_d)$.

Regular embeddings (jackets)

A cellular embedding $\phi : |\Gamma| \to F$ of a $(d+1)$-colored graph (Γ, γ) into a (closed) surface F is called a regular embedding if there exists a cyclic permutation $\varepsilon = (\varepsilon_0, \dots, \varepsilon_d)$ of Δ_d s.t. each connected component of $F - \phi(|\Gamma|)$ is an open ball bounded by the image of an $\{\varepsilon_i, \varepsilon_{i+1}\}-$ colored cycle of Γ $(\forall i \in \mathbb{Z}_d)$.

EXAMPLE: Regular embedding corresponding to $\varepsilon =$ (green, red, blue, grey)



The regular genus

Gagliardi, 1981

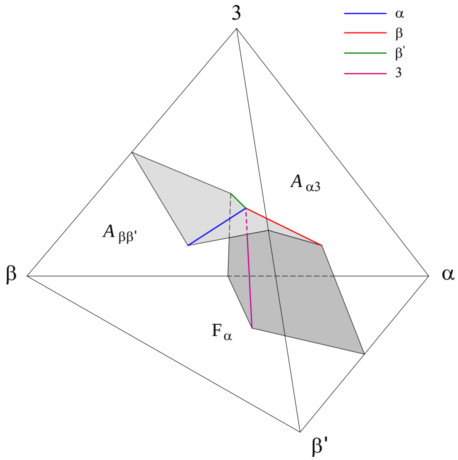
For each $(d+1)$-colored graph (Γ, γ) and for every cyclic permutation ε of Δ_d, there exists a *regular embedding* of Γ onto a suitable surface F_ε. Moreover:

- F_ε is orientable if and only if Γ is bipartite;
- ε and ε^{-1} induce the same embedding.

Definition

The *regular genus* $\rho_\epsilon(\Gamma)$ of Γ with respect to ε is the classical genus (resp. half of the genus) of the orientable (resp. non-orientable) surface F_ε :

$$\sum_{i \in \mathbb{Z}_{d+1}} g_{\varepsilon_i \varepsilon_{i+1}} + (1-d)p = 2 - 2\rho_\varepsilon(\Gamma)$$



$$\varepsilon = (\beta', \alpha, \beta, 3)$$

Regular embeddings and Gurau degree

Regular genus of Γ

$$\rho(\Gamma) = \min \{\rho_\varepsilon(\Gamma) \ / \ \ \varepsilon \text{ cyclic permutation of } \Delta_d\}$$

Gurau degree

Given a $(d+1)$-colored graph (Γ, γ), then

$$\omega_G(\Gamma) = \sum_{i=1}^{\frac{d!}{2}} \rho_{\varepsilon^{(i)}}(\Gamma)$$

where the $\varepsilon^{(i)}$'s are the cyclic permutations of Δ_d up to inverse.
Hence,

$$\omega_G(\Gamma) \geq \frac{d!}{2} \rho(\Gamma)$$

PL invariants

Definition

- regular genus of a singular d-manifold N^d:

$$\mathcal{G}(N^d) = \min \{\rho(\Gamma) \mid (\Gamma,\gamma) \text{ represents } N^d\}$$

- Gurau degree (G-degree) of a singular d-manifold N^d:

$$\mathcal{D}_G(N^d) = \min \{\omega_G(\Gamma) \mid (\Gamma,\gamma) \text{ represents } N^d\}$$

Remark: The minimum is always realized by a contracted graph.

$$\mathcal{D}_G(\Gamma) \geq \frac{d!}{2}\mathcal{G}(\Gamma)$$

Problems and questions

- determine properties of the G-degree for graphs and/or pseudomanifolds

Problems and questions

- determine properties of the G-degree for graphs and/or pseudomanifolds
- TOP and/or PL classification of $N^d = |K(\Gamma)|$ according to $\omega_G(\Gamma)$ and $\mathcal{D}_G(N^d)$

Problems and questions

- determine properties of the G-degree for graphs and/or pseudomanifolds
- TOP and/or PL classification of $N^d = |K(\Gamma)|$ according to $\omega_G(\Gamma)$ and $\mathcal{D}_G(N^d)$
- determine topological/geometric/combinatorial properties of $K(\Gamma)$ from $\omega_G(\Gamma)$

Problems and questions

- determine properties of the G-degree for graphs and/or pseudomanifolds
- TOP and/or PL classification of $N^d = |K(\Gamma)|$ according to $\omega_G(\Gamma)$ and $\mathcal{D}_G(N^d)$
- determine topological/geometric/combinatorial properties of $K(\Gamma)$ from $\omega_G(\Gamma)$
- find relations of $\mathcal{D}_G$ with other PL invariants computed on colored graphs

Problems and questions

- determine properties of the G-degree for graphs and/or pseudomanifolds
- TOP and/or PL classification of $N^d = |K(\Gamma)|$ according to $\omega_G(\Gamma)$ and $\mathcal{D}_G(N^d)$
- determine topological/geometric/combinatorial properties of $K(\Gamma)$ from $\omega_G(\Gamma)$
- find relations of $\mathcal{D}_G$ with other PL invariants computed on colored graphs
- in the $\frac{1}{N}$-expansion:

Problems and questions

- determine properties of the G-degree for graphs and/or pseudomanifolds
- TOP and/or PL classification of $N^d = |K(\Gamma)|$ according to $\omega_G(\Gamma)$ and $\mathcal{D}_G(N^d)$
- determine topological/geometric/combinatorial properties of $K(\Gamma)$ from $\omega_G(\Gamma)$
- find relations of $\mathcal{D}_G$ with other PL invariants computed on colored graphs
- in the $\frac{1}{N}$-expansion:
  - which terms of the sequence do actually appear ?
  - which kind of exponents correspond to closed/singular manifolds ?
  - which (classes of) pseudomanifolds do actually appear in certain terms of the sequence ?

Which terms do appear in the $\frac{1}{N}$-expansion ?

Casali - Grasselli, 2018

For d even and $d \geq 4$, if Γ is bipartite or Γ represents a singular d-manifold, then

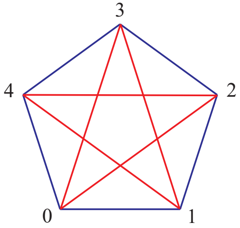
$$\omega_G(\Gamma) \equiv 0 \mod (d-1)!$$

Consequences: In the case of d even and $d \geq 4$, the only non-null terms of the $\frac{1}{N}$-expansion are the ones corresponding to even (integer) powers of $\frac{1}{N}$.

Hint of proof:

Bijective correspondence

cyclic permutation of Δ_d $\longleftrightarrow$ Hamiltonian cycle of K_{d+1}



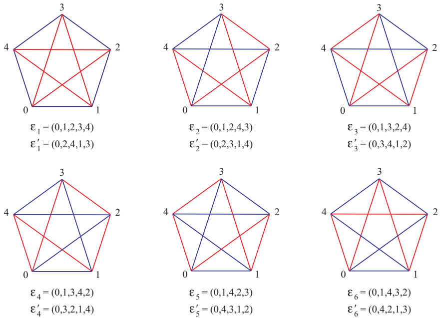
$\varepsilon_1 = (0,1,2,3,4)$

$\varepsilon'_1 = (0,2,4,1,3)$

Example: two Hamiltonian cycles of K_5 and their associated cyclic permutations

Bryant - Horsley - Maenhaut - Smith, 2011

For all odd $n \geq 3$ there exists a partition of all Hamiltonian cycles of K_n (the complete graph with n vertices) into $(n-2)!$ Hamiltonian cycle decompositions of K_n.



Example: the six Hamiltonian cycle decompositions of K_5

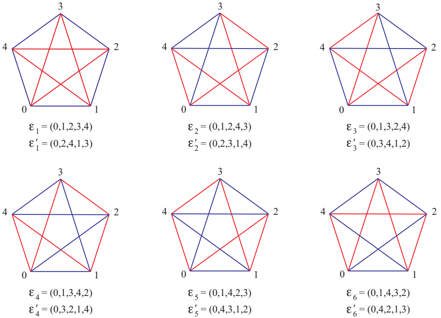
Consequence: The number of bicolored cycles of Γ can be computed by using only the permutations belonging to one class.

$$\omega_G(\Gamma) = (d-1)! \cdot \sum_{i=1}^{d/2} \rho_{\bar{\varepsilon}^{(i)}}(\Gamma)$$

where $\bar{\varepsilon}^{(1)}, \bar{\varepsilon}^{(2)}, \ldots, \bar{\varepsilon}^{(\frac{d}{2})}$ form one of the partition classes of the cyclic permutations of Δ_d

Remark: If d is odd, the sum $\sum_{i=1}^{d/2} \rho_{\bar{\varepsilon}^{(i)}}$ is still constant on each class and it is $\frac{2}{(d-1)!}\omega_G(\Gamma)$.

Dimension 4



For any 5-colored graph Γ,

$$\omega_G(\Gamma) = 6(\rho_\varepsilon(\Gamma) + \rho_{\varepsilon'}(\Gamma))$$

for each pair of associated permutations ε and ε'.

First terms of the $\frac{1}{N}$-expansion

Casali - C. - Dartois - Grasselli, 2018

(Γ, γ) any bipartite $(d+1)$-colored graph.

$$\omega_G(\Gamma) < \frac{d!}{2} \implies |K(\Gamma)| \cong_{PL} \mathbb{S}^d$$

Consequences: In the $\frac{1}{N}$-expansion the terms with exponents $> -d$ count only graphs representing $\mathbb{S}^d$

The non-bipartite case

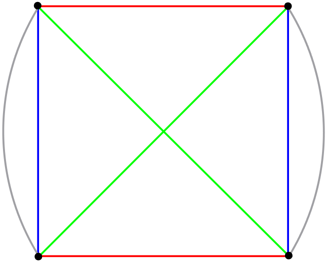
Casali - C. - Dartois - Grasselli, 2018

For $d \geq 3$, no non-bipartite $(d+1)$-colored graph (Γ, γ) exists with $\omega_G(\Gamma) < \lceil \frac{d!}{4} \rceil$.

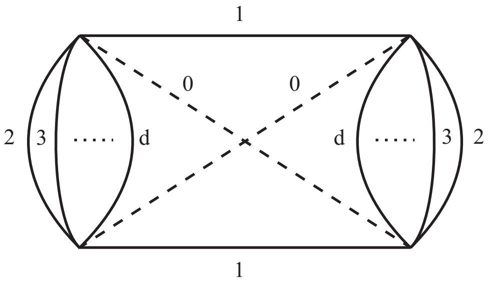
Moreover, for each $d \geq 4$ no colored graph representing a singular d-manifold exists with G-degree $< \lceil \frac{d!}{2} \rceil$.

Remark: For $d = 3$ the bound is sharp.

Example: Γ representing $\Sigma(\mathbb{RP}^2)$ with $\omega_G(\Gamma) = 2$



Generalization: Γ representing $\Sigma^{(d-2)}(\mathbb{RP}^2)$ with $\omega_G(\Gamma) = \frac{(d-1)!}{2}(d-1)$



The non-bipartite case

Bonzom - Lionni - Tanasa, 2018

For $d \geq 3$, no non-bipartite $(d+1)$-colored graph (Γ, γ) exists with $\omega_G(\Gamma) < \frac{(d-1)!(d-1)}{2}$.

Finiteness property

Casali - C. - Grasselli, 2018

For each pair of fixed integers $\bar{S} \geq 0$ and $\bar{R} \geq d+1$, only a finite number of $(d+1)$-colored graphs (Γ, γ) exists, with $\omega_G(\Gamma) = \bar{S}$ and $\sum_{i \in \Delta_d} g_{\hat{i}} = \bar{R}$.

Hence, the G-degree $\mathcal{D}_G$ is finite-to-one within the set all singular d-manifolds with $h \geq 0$ singular vertices (in particular within the set of closed manifolds).

Dimension 3

The gem-complexity of a singular d-manifold N^d is

$$k(N^d) = p - 1$$

where $2p$ = minimum order of a $(d+1)$-colored graph representing N^d

Casali - C. - Dartois - Grasselli, 2018

- For any 4-colored graph (Γ, γ) of order $2p$:

$$\omega_G(\Gamma) = p - 1 - \sum_{i \in \Delta_3} (g_{\hat{i}} - 1) + \chi(K(\Gamma))$$

- For any closed 3-manifold M

$$\mathcal{D}_G(M) = k(M)$$

Casali - C. - Grasselli 2018

For each singular 3-manifold N^3 with only one singular point,

$$\mathcal{D}_G(N^3) = k(N^3) + g(\partial \check{N}^3)$$

Conjecture

Let N^3 be a singular 3-manifold and let $\partial \check{N}_1, \dots, \partial \check{N}_h$ be the boundary components of $\check{N}^3$. Then:

$$\mathcal{D}_G(N^3) = k(N^3) + \sum_{i=1}^{h} g(\partial \check{N}_i)$$

Notation: S = closed surface, $g(S) = \begin{cases} genus(S) & \text{if } S \text{ is orientable} \\ \frac{1}{2} genus(S) & \text{if } S \text{ is non-orientable} \end{cases}$

Classification according to $\mathcal{D}_G$

For prime, handle-free orientable (resp. non-orientable) closed 3-manifolds the classification is complete up to

$$\mathcal{D}_G(M) = 15 \quad (\text{resp. } \mathcal{D}_G(M) = 14)$$

Classification according to $\mathcal{D}_G$

For prime, handle-free orientable (resp. non-orientable) closed 3-manifolds the classification is complete up to

$$\mathcal{D}_G(M) = 15 \quad (\text{resp. } \mathcal{D}_G(M) = 14)$$

Let M be a closed, orientable, prime 3-manifold, then,

- $\mathcal{D}_G(M) \leq 10 \Rightarrow M \cong \mathbb{S}^2 \times \mathbb{S}^1$ or it is spherical
- $\mathcal{D}_G(M) \leq 13 \Rightarrow M$ is not hyperbolic ($\mathcal{D}_G(M) \leq 11$ $\Rightarrow\ M \cong \mathbb{S}^2 \times \mathbb{S}^1$ or M is flat or spherical)
- The Matveev-Fomenko-Weeks manifold has G-degree $= 14$

Classification according to ω_G (bipartite case)

Casali - C. - Dartois - Grasselli 2018; Casali - C. - Grasselli 2018

Let (Γ, γ) be a bipartite 4-colored graph and $N^3 = |K(\Gamma)|$, then

$$\begin{array}{rcl}
\omega_G(\Gamma) \leq 2 & \Longrightarrow & N^3 \cong \mathbb{S}^3 \\
\omega_G(\Gamma) = 3 & \Longrightarrow & \check{N}^3 \cong \mathbb{Y}^3_1 \\
\omega_G(\Gamma) \in \{4,5\} & \Longrightarrow & \text{either} \quad N^3 \cong \mathbb{S}^3, \mathbb{S}^2 \times \mathbb{S}^1, \mathbb{RP}^3, L(3,1) \\
 & & \text{or} \quad \check{N}^3 \cong \mathbb{Y}^3_1,\ \mathbb{T}^2 \times I \\
\omega_G(\Gamma) = 6 & \Longrightarrow & \text{either} \quad N^3 \cong \mathbb{S}^3, \mathbb{S}^2 \times \mathbb{S}^1, \mathbb{RP}^3, L(3,1), \mathbb{RP}^3 \# \mathbb{RP}^3 \\
 & & \text{or} \quad \check{N}^3 \cong \mathbb{Y}^3_1,\ \mathbb{T}^2 \times I,\ \mathbb{D}^2_2 \times \mathbb{S}^1,\ \mathbb{Y}^3_1 \# \mathbb{Y}^3_1,\ M_S, \\
 & & (\mathbb{S}^1 \times \mathbb{S}^2) \# \mathbb{Y}^3_1,\ \mathbb{RP}^3 \# \mathbb{Y}^3_1,\ \mathbb{Y}^3_2
\end{array}$$

$\mathbb{Y}^d_g$ = d-dimensional orientable genus g handlebody

$\mathbb{T}^2$ = torus, $\mathbb{D}^2_2$ = disk with two holes

M_S = Seifert manifold with base $\mathbb{D}^2$ and Seifert parameters $((2,1),(2,1))$

Classification in dimension 3 and 4

n=3

- TOP=PL (any topological 3-manifold admits a PL-structure which is unique up to PL-isomorphisms)
- PL=DIFF (each PL-structure on a 3-manifold is smoothable in a unique way up to diffeomorphisms)

n=4

- PL=DIFF
- TOP≠PL
  - there are topological 4-manifolds admitting no PL structure;
  - there can be infinitely many non-equivalent PL structures on the same topological 4-manifold.

[Akhmedov-Doug Park, 2010], [Akhmedov-Ishida-Doug Park, 2013]

There exist (infinitely many) non-equivalent PL structures on:

- $\#_{2h-1}\mathbb{CP}^2\#_{2h}\overline{\mathbb{CP}^2}$, for any integer $h \geq 1$
- $\#_{2h-1}(\mathbb{S}^2 \times \mathbb{S}^2)$, for $h \geq 138$
- $\#_{2h-1}(\mathbb{CP}^2\#\overline{\mathbb{CP}^2})$, for $h \geq 23$
- $\#_{2p}(\mathbb{S}^2 \times \mathbb{S}^2)$ and $\#_{2p}(\mathbb{CP}^2\#\overline{\mathbb{CP}^2})$, for large enough integers p not divisible by 4.

Dimension 4

Casali - C. - Dartois - Grasselli, 2018

- For any 5-colored graph (Γ, γ) of order $2p$ such that $|K(\Gamma)|$ is a singular 4-manifold:

$$\omega_G(\Gamma) = 6(p - 1 - \sum_{i \in \Delta_4} (g_{\hat{i}} - 1) + \chi(K(\Gamma)) - 2)$$

- For any closed 4-manifold M:

$$\mathcal{D}_G(M) = 6(k(M) + \chi(M) - 2)$$

Dimension 4

Casali - C. - Dartois - Grasselli, 2018

- For any 5-colored graph (Γ, γ) of order $2p$ representing a singular 4-manifold:

$$\omega_G(\Gamma) = 6(p - 1 - \sum_{i \in \Delta_4} (g_{\hat{i}} - 1) + \chi(K(\Gamma)) - 2)$$

- For any closed 4-manifold M:

$$\mathcal{D}_G(M) = 6(\underbrace{k(M)}_{\text{PL}} + \underbrace{\chi(M)}_{\text{TOP}} - 2)$$

Effective computation of $\mathcal{D}_G$

Casali - C. - Dartois - Grasselli, 2018

If M is PL-homeomorphic to

$$(\#_p\mathbb{CP}^2)\#(\#_{p'}(\overline{\mathbb{CP}^2}))\#(\#_q\mathbb{S}^2\times\mathbb{S}^2)\#(\#_r\mathbb{S}^3\otimes\mathbb{S}^1)\#(\#_s\mathbb{RP}^4)\#(\#_t K3)$$

with $p, p', q, r, s, t \geq 0$. Then,

$$\mathcal{D}_G(M) = 12 \cdot \left[2(p + p' + 2q + 22t) + r + 3s\right]$$

$\mathbb{S}^3 \otimes \mathbb{S}^1 =$ orientable or nonorientable $\mathbb{S}^3$-bundle over $\mathbb{S}^1$

It is always realized by simple/semisimple contracted graphs

TOP classification

Casali - C. - Dartois - Grasselli, 2018

Let (Γ, γ) be a gem of a simply-connected closed PL 4-manifold M. If $\omega_G(\Gamma) \leq 527$, then M is TOP-homeomorphic to

$$(\#_r \mathbb{CP}^2)\#(\#_{r'}(\overline{\mathbb{CP}^2})) \quad \textit{or} \quad \#_s(\mathbb{S}^2 \times \mathbb{S}^2)$$

where $r + r' = \beta_2(M)$ and $s = \frac{1}{2}\beta_2(M)$, with $\beta_2(M) \leq \frac{1}{24} \cdot \omega_G(\Gamma)$

PL classification

Casali - C. - Dartois - Grasselli 2018, Casali preprint

If (Γ, γ) is a 5-colored graph representing a singular 4-manifold N^4, then:

$$\omega_G(\Gamma) \in \{0, 6\} \implies N^4 \cong \mathbb{S}^4$$

$$\omega_G(\Gamma) = 12 \implies \text{either} \quad N^4 \in \{\mathbb{S}^4,\ \mathbb{S}^3 \times \mathbb{S}^1,\ \mathbb{S}^3 \tilde{\times} \mathbb{S}^1\}$$
$$\text{or} \quad \check{N}^4 \in \{\mathbb{Y}^4_1,\ \tilde{\mathbb{Y}}^4_1\}$$

$$\omega_G(\Gamma) = 18 \implies \textit{either} \ N^4 \in \{\mathbb{S}^4,\ \mathbb{S}^3 \times \mathbb{S}^1,\ \mathbb{S}^3 \tilde{\times} \mathbb{S}^1\}$$
$$\text{or} \quad \check{N}^4 \in \{\mathbb{Y}^4_1,\ \tilde{\mathbb{Y}}^4_1,\ \mathbb{RP}^3 \times I,\ (\mathbb{S}^1 \times \mathbb{S}^2) \times I,\ (\mathbb{S}^1 \tilde{\times} \mathbb{S}^2) \times I\}$$

PL classification

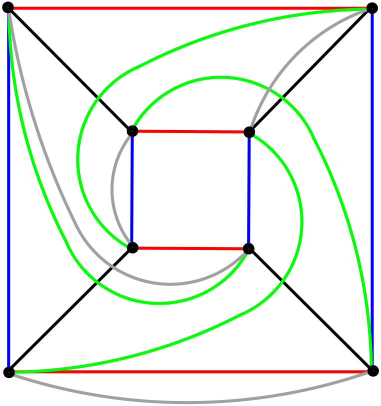
Moreover, if Γ has exactly one singular color and $\omega_G(\Gamma) = 24$. then:

either $N^4 \in \{\mathbb{S}^4,\ \mathbb{S}^3 \times \mathbb{S}^1,\ \mathbb{S}^3 \tilde{\times} \mathbb{S}^1,\ \#_2(\mathbb{S}^1 \times \mathbb{S}^3),\ \#_2(\mathbb{S}^1 \tilde{\times} \mathbb{S}^3),\ \mathbb{CP}^2\}$

or $\check{N}^4 \in \{\mathbb{Y}^4_1,\ \tilde{\mathbb{Y}}^4_1,\ \mathbb{Y}^4_1 \# \mathbb{Y}^4_1,\ \tilde{\mathbb{Y}}^4_1 \# \tilde{\mathbb{Y}}^4_1,\ \mathbb{Y}^4_2,\ \tilde{\mathbb{Y}}^4_2,\ \mathbb{Y}^4_1 \# (\mathbb{S}^1 \times \mathbb{S}^3),\ \tilde{\mathbb{Y}}^4_1 \# (\mathbb{S}^1 \times \mathbb{S}^3),\ \mathbb{S}^2 \times \mathbb{D}^2,\ \xi_2\}$

$\xi_2 = \mathbb{D}^2$-bundle over $\mathbb{S}^2$ with Euler number 2

ξ_2



PL classification according to $\mathcal{D}_G$

Casali - C. - Dartois - Grasselli 2018, Casali preprint

Let N^4 be a singular 4-manifold. Then:

$$\mathcal{D}_G(N^4) = 0 \iff N^4 \cong \mathbb{S}^4$$

$$\mathcal{D}_G(N^4) = 12 \iff \text{either } N^4 \in \{\mathbb{S}^3 \times \mathbb{S}^1,\ \mathbb{S}^3 \tilde{\times} \mathbb{S}^1\} \text{ or } \check{N}^4 \in \{\mathbb{Y}^4_1,\ \tilde{\mathbb{Y}}^4_1\}$$

$$\mathcal{D}_G(N^4) = 18 \iff \check{N}^4 \in \{L(2,1) \times I,\ (\mathbb{S}^1 \times \mathbb{S}^2) \times I,\ (\mathbb{S}^1 \tilde{\times} \mathbb{S}^2) \times I\}$$

If N^4 has at most one singular vertex, then:

$$\mathcal{D}_G(N^4) = 24 \iff \text{either } N^4 \in \{\#_2(\mathbb{S}^1 \times \mathbb{S}^3),\ \#_2(\mathbb{S}^1 \tilde{\times} \mathbb{S}^3),\ \mathbb{CP}^2\} \text{ or } \check{N}^4 \in \{\mathbb{Y}^4_2,\ \tilde{\mathbb{Y}}^4_2,\ \mathbb{Y}^4_1 \# (\mathbb{S}^1 \times \mathbb{S}^3),\ \tilde{\mathbb{Y}}^4_1 \# (\mathbb{S}^1 \times \mathbb{S}^3),\ \mathbb{S}^2 \times \mathbb{D}^2,\ \xi_2\}$$

If N^4 is a closed simply-connected 4-manifold, then

$$\mathcal{D}_G(M) = 48 \quad \Longleftrightarrow \quad M \cong \mathbb{S}^2 \times \mathbb{S}^2, \mathbb{CP}^2 \# \mathbb{CP}^2, \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$$

and no other closed simply-connected PL 4-manifold exists with G-degree ≤ 59.

HINT OF PROOFS:

In all these cases $K(\Gamma)$ may be assumed to have $d+1$ vertices.

If (Γ, γ) is such a graph, N^4 is a closed manifold and $\{r, s, t\} \cup \{i, j\} = \Delta_5$:

$$N^4 = \underbrace{\mathbb{D}^4 \cup (H_1^{(1)} \cup \cdots \cup H_{r_1}^{(1)}) \cup (H_1^{(2)} \cup \cdots \cup H_{r_2}^{(2)})}_{N(r,s,t)} \cup \underbrace{(H_1^{(3)} \cup \cdots \cup H_{r_3}^{(3)}) \cup H^{(4)}}_{N(i,j)}$$

where:

- $N(r, s, t)$ = regular neighborhood of the subcomplex of $K(\Gamma)$ generated by vertices labelled by $\{r, s, t\}$ *(union of 0,1,2-handles)*
- $N(i, j)$ = regular neighborhood of the subcomplex of $K(\Gamma)$ generated by vertices labelled by $\{i, j\}$ *(union of 3,4-handles)*

HINT OF PROOFS:

- no 2-handles $\Rightarrow$ $N^4 \cong \#_m(\mathbb{S}^1 \times \mathbb{S}^3)$
  (via [Montesinos, 1979])

HINT OF PROOFS:

- no 2-handles $\Rightarrow$ $N^4 \cong \#_m(\mathbb{S}^1 \times \mathbb{S}^3)$
  (via [Montesinos, 1979])
- one 2-handle and $\partial N(r,s,t) \cong \mathbb{S}^3$ $\Rightarrow$ $N^4 \cong \mathbb{CP}^2$
  (via [Gordon-Luecke, 1989], ensuring $(K,d) = (K_0, 0)$)
- one 2-handle and $\partial N(r,s,t) \cong \mathbb{S}^2 \times \mathbb{S}^1$ $\Rightarrow$ $\check{N}^4 \cong \mathbb{S}^2 \times \mathbb{D}^2$
  (via [Gabai, 1987], ensuring $(K,d) = (K_0, 1)$)
- one 2-handle and $\partial N(r,s,t) \cong \mathbb{RP}^3$ $\Rightarrow$ $\check{N}^4 \cong \xi_2$
  (via [Kronheimer-Mrowka-Ozsvath-Szabo, 2007], ensuring $(K,d) = (K_0, 2)$)

Additivity of G-degree

$\mathcal{D}_G$ is additive on connected sums of

$$\mathbb{S}^4,\ \mathbb{CP}^2,\ \mathbb{S}^2\times\mathbb{S}^2,\ \mathbb{S}^3\times\mathbb{S}^1,\ \mathbb{S}^3\tilde{\times}\mathbb{S}^1,\ \mathbb{RP}^4,\ K3$$

Additivity of G-degree

$\mathcal{D}_G$ is additive on connected sums of

$$\mathbb{S}^4,\ \mathbb{CP}^2,\ \mathbb{S}^2\times\mathbb{S}^2,\ \mathbb{S}^3\times\mathbb{S}^1,\ \mathbb{S}^3\tilde{\times}\mathbb{S}^1,\ \mathbb{RP}^4,\ K3$$

Exotic structures and $\mathcal{D}_G$

$\mathcal{D}_G$ does not satisfy the additivity property, within the set of closed PL 4-manifolds.

Example: let N and N' be two of the infinitely many different PL manifolds homeomorphic to $\mathbb{CP}^2\#(\#_2(-\mathbb{CP}^2))$.

By Wall theorem and additivity:

$\mathcal{D}_G(N)=\mathcal{D}_G(N') \quad \Longrightarrow \quad$ impossible by finiteness property of $\mathcal{D}_G$.

THANK YOU