Geometric and spectral aspects of planar maps with high degrees

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ERC CombiTop

Joint with
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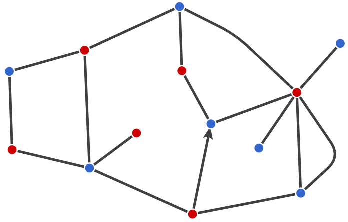
Boltzmann planar maps

- A (planar) map $\mathfrak{m}$ is a finite connected (multi-)graph embedded in the sphere viewed up to homeomorphisms. We assume it to be rooted and bipartite.



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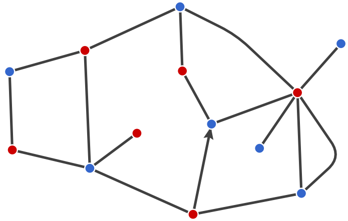

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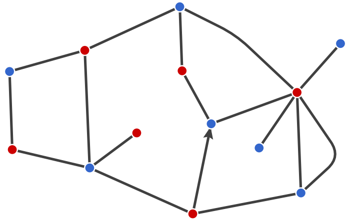

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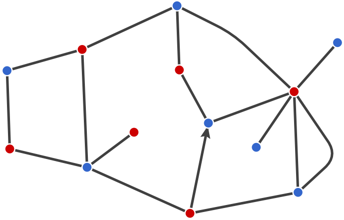
$\mathbf{P}_{\mathbf{q}}\Bigg($  $\Bigg) = \frac{1}{Z_{\mathbf{q}}} \cdot q_1 \cdot q_2^2 \cdot q_3 \cdot q_4 \cdot q_5$

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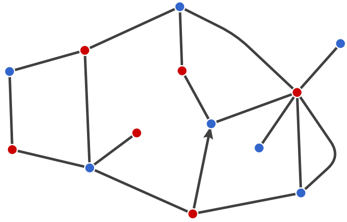
$\mathbf{P_q}\Big($  $\Big) = \frac{1}{Z_\mathbf{q}} \cdot q_1 \cdot q_2^2 \cdot q_3 \cdot q_4 \cdot q_5$

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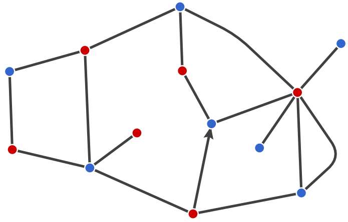
  - $q_k \sim \mathrm{p}_\mathbf{q} \cdot c_\mathbf{q}^{-k+1} \cdot k^{-a}$ with $a \in (\frac{3}{2}, \frac{5}{2})$ and $\mathrm{p}_\mathbf{q}, c_\mathbf{q} > 0$ fine tuned so $\mathbf{q}$ is critical. Example for $a = 2$: $q_k = (6^{k-1}(2k-1)(2k-3))^{-1}\mathbf{1}_{k\geq 2}$.

$\mathbf{P_q}\Bigg($  $\Bigg) = \frac{1}{Z_\mathbf{q}} \cdot q_1 \cdot q_2^2 \cdot q_3 \cdot q_4 \cdot q_5$

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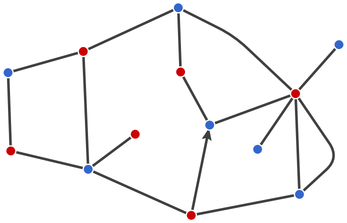
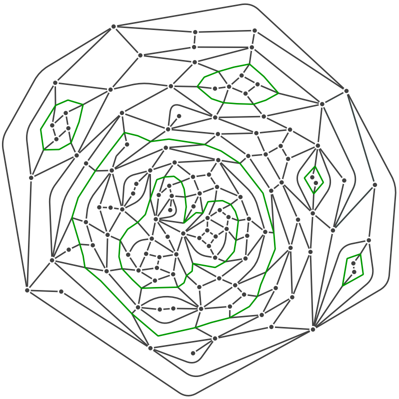
    Example for $a = 2$: $q_k = (6^{k-1}(2k-1)(2k-3))^{-1}\mathbf{1}_{k \geq 2}$.
    For a typical face: $\mathbf{P}(\deg \geq k) \sim C \cdot k^{-(a-1/2)}$.

$\mathbf{P}_{\mathbf{q}}\Bigg($  $\Bigg) = \frac{1}{Z_{\mathbf{q}}} \cdot q_1 \cdot q_2^2 \cdot q_3 \cdot q_4 \cdot q_5$

Boltzmann maps as the gasket of a loop-decorated quadrangulation



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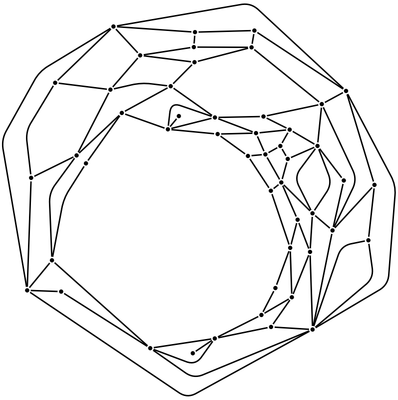
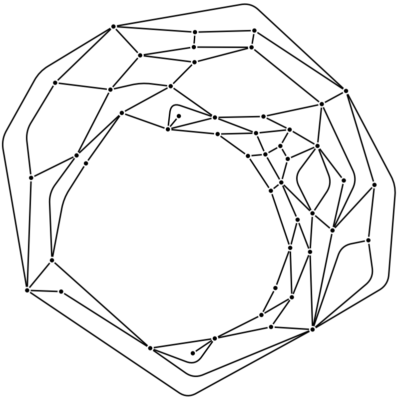


Boltzmann maps as the gasket of a loop-decorated quadrangulation

critical $O(n)$ loop model $\iff$ critical Boltzmann map with

$$\begin{cases} a = 2 - \frac{1}{\pi}\arccos(\frac{n}{2}) \text{ in the } \textit{dense} \text{ phase,} \\ a = 2 + \frac{1}{\pi}\arccos(\frac{n}{2}) \text{ in the } \textit{dilute} \text{ phase.} \end{cases}$$

Supposedly related to γ-LQG with $\gamma = \min(2\sqrt{a-1}, 2/\sqrt{a-1}) \in (\sqrt{2}, 2)$.



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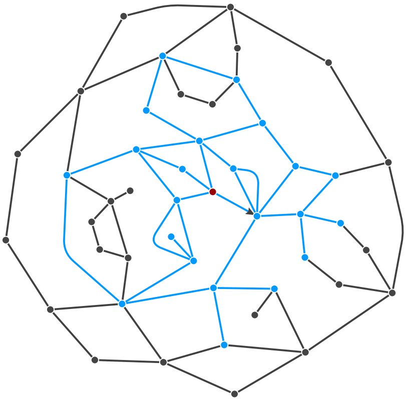

Infinite Boltzmann planar maps

- Sample $\mathfrak{M}_n$ from $\mathbf{P_q}(\cdot \mid n \text{ vertices})$, then $\mathfrak{M}_n \to \mathfrak{M}_\infty$ in distribution in the *local limit* sense: $\mathbf{P}(\mathrm{Ball}_r(\mathfrak{M}_n) = \mathfrak{m}) \to \mathbf{P}(\mathrm{Ball}_r(\mathfrak{M}_\infty) = \mathfrak{m})$ for each $r \in \mathbf{N}$ and each map $\mathfrak{m}$. [Stephenson '15, also Björnberg & Stefánsson '14]

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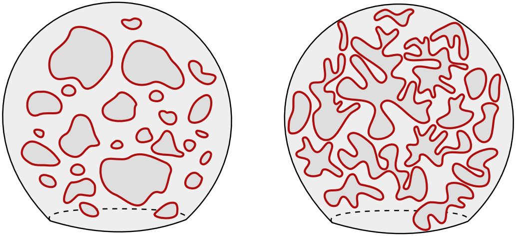

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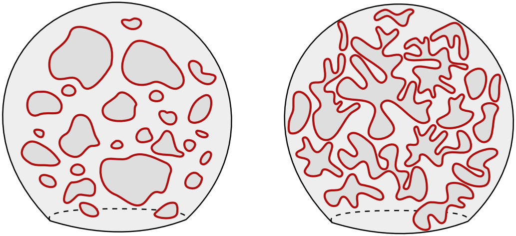

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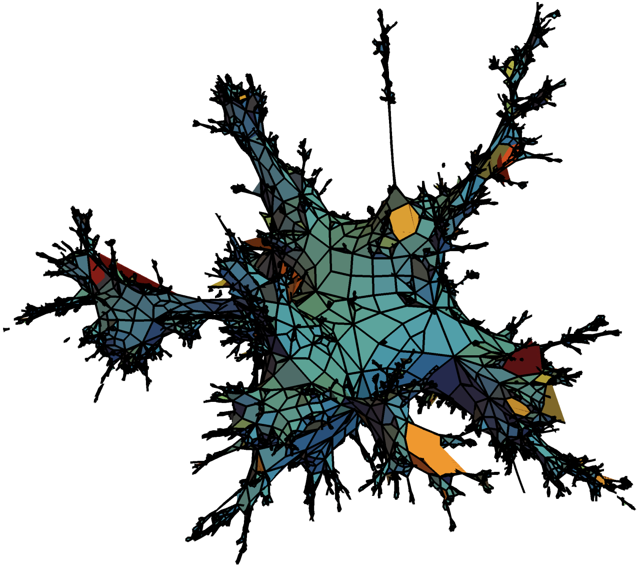
- Idea: study the *dual* map $\mathfrak{M}_\infty^\dagger$! The volume $|\mathrm{Ball}_r(\mathfrak{M}_\infty^\dagger)|$ should increase much faster when $a < 2$ than when $a > 2$.

Simulation (by N. Curien): $\mathfrak{M}_\infty$ for $a = 2.45$



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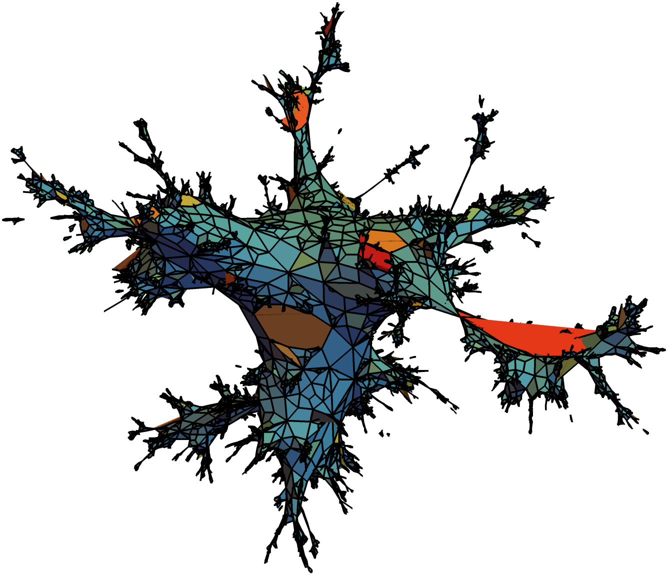

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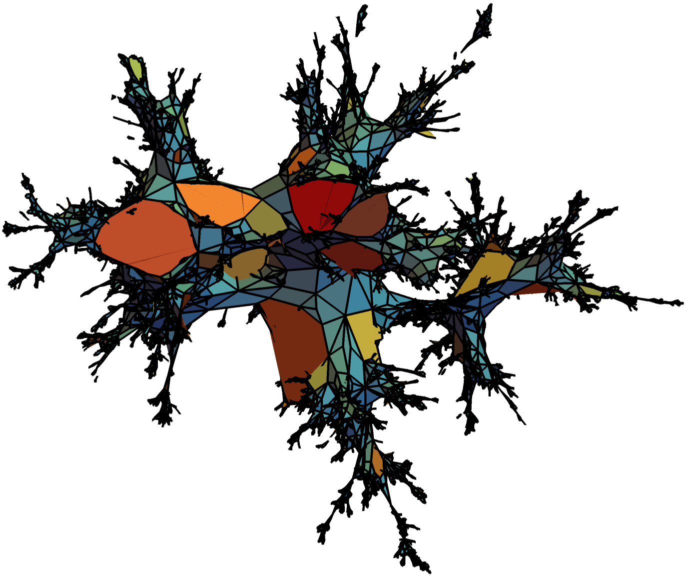

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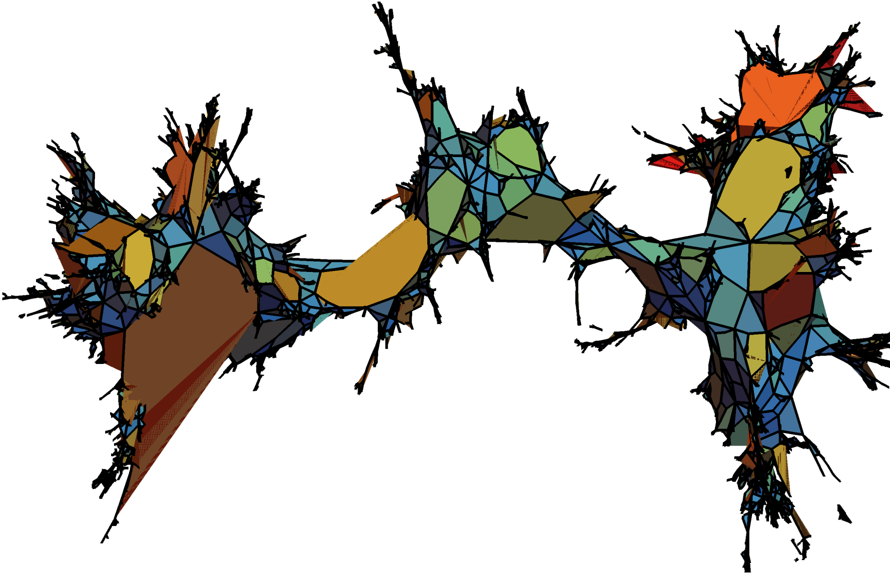

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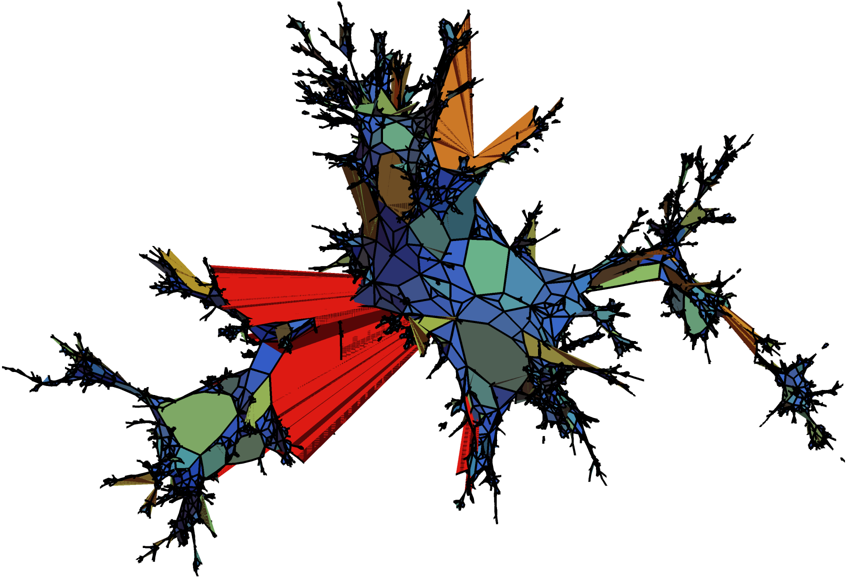
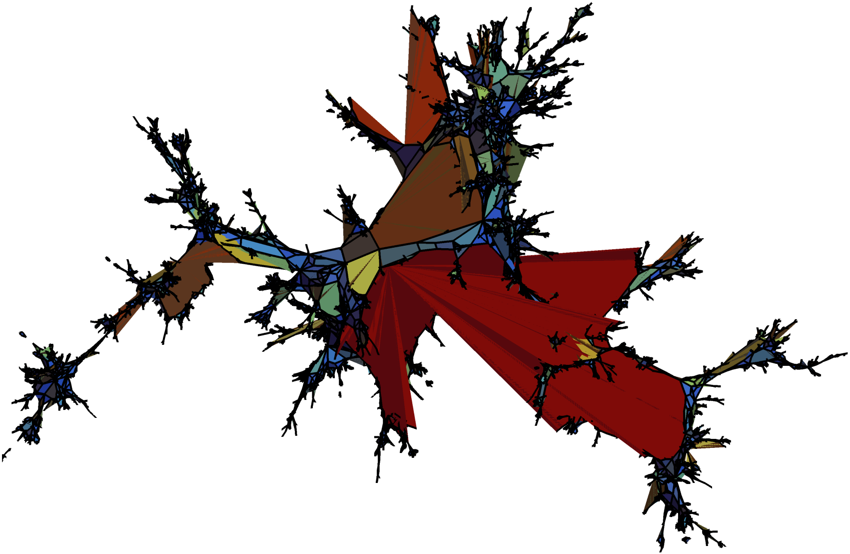
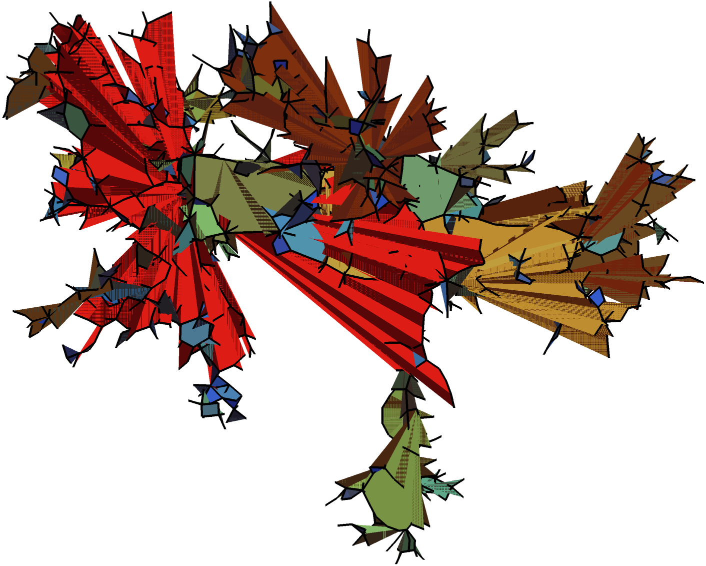


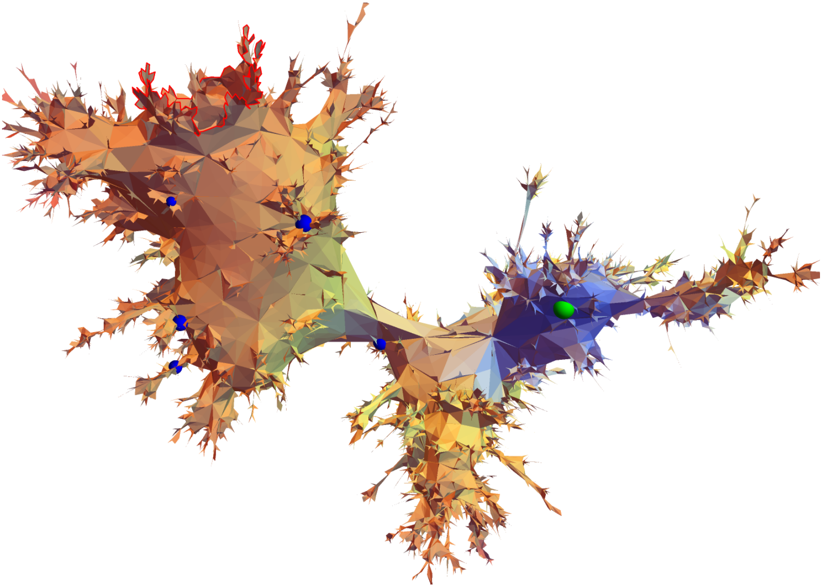
Simulation (by N. Curien): $\mathfrak{M}_\infty$ for $a = 1.9$



Simulation (by N. Curien): $\mathfrak{M}_\infty$ for $a = 1.7$



Simulation (by T. Budd): $\mathfrak{M}_\infty^\dagger$ for $a = 2.45$



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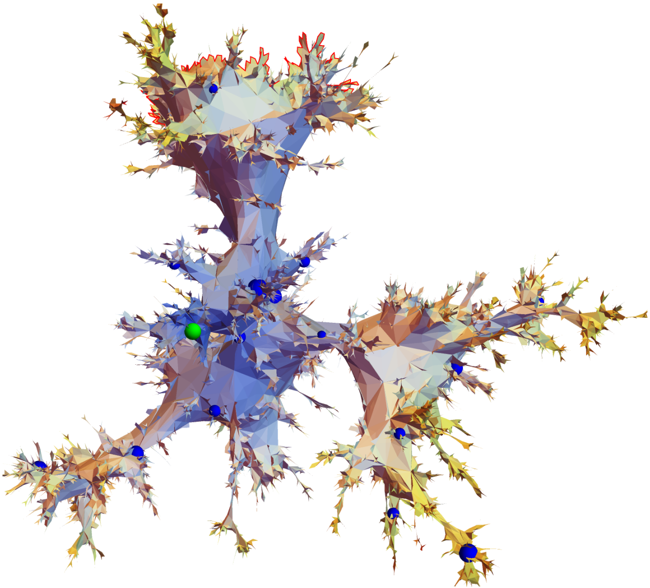

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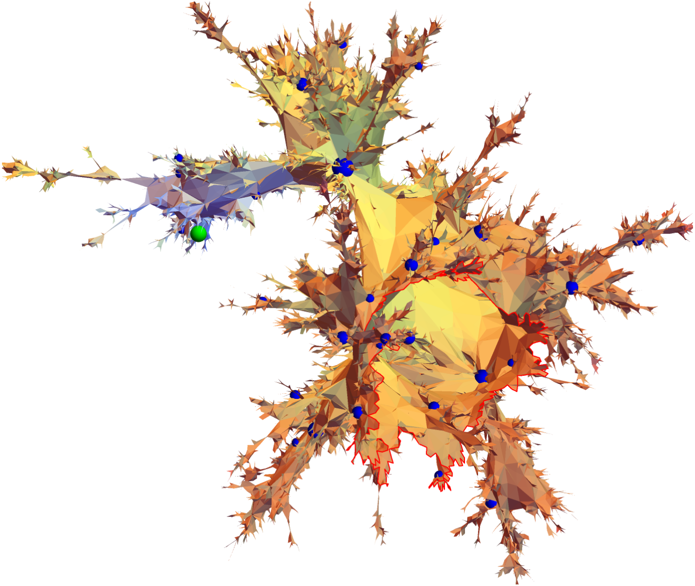

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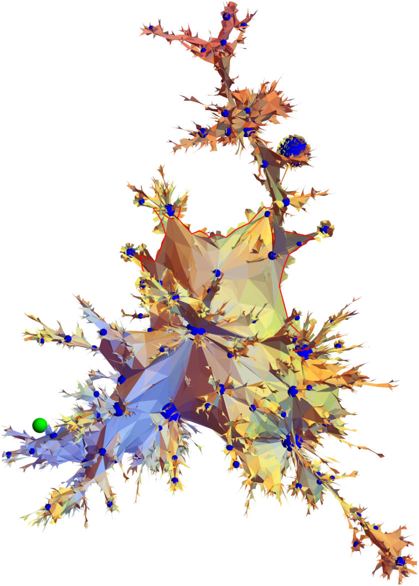

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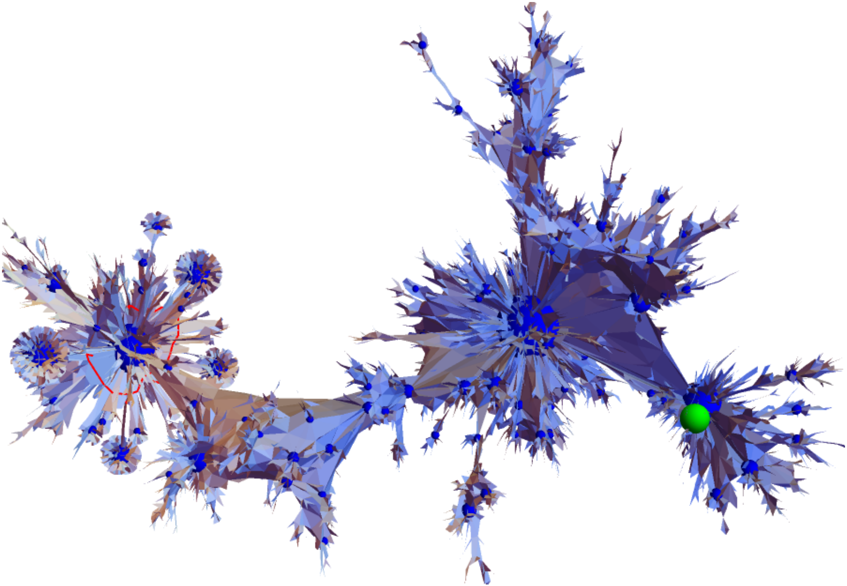

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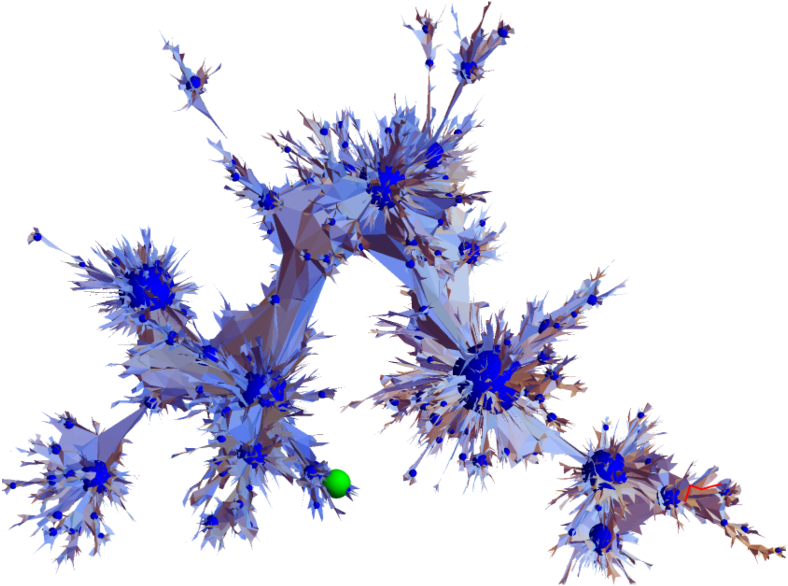

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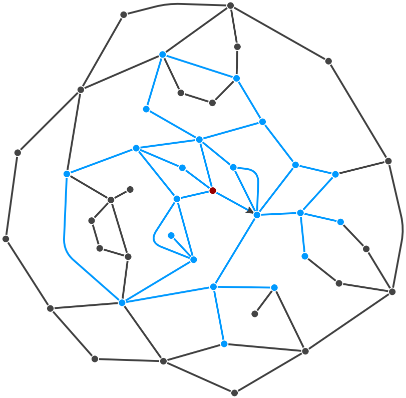

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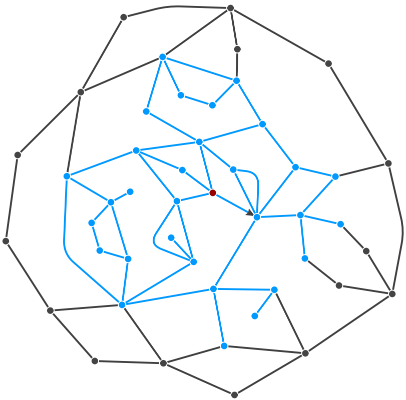

Volume growth

$$q_k \sim p_{\mathbf{q}} \cdot c_{\mathbf{q}}^{-k+1} \cdot k^{-a}$$

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In the sense:

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- If $a < 2$, then $r^{-1} \log \lvert\overline{\mathrm{Ball}}_r(\mathfrak{M}_\infty^\dagger)\rvert \to C > 0$ in probability.

THEOREM (Curien & ☺ in progress). For every $a \in (\frac{3}{2}, \frac{5}{2})$, it holds that $\lvert\mathrm{Ball}_r(\mathfrak{M}_\infty)\rvert \approx \lvert\overline{\mathrm{Ball}}_r(\mathfrak{M}_\infty)\rvert \approx r^{2a-1}$.

Volume growth

$$q_k \sim \mathrm{p}_{\mathbf{q}} \cdot c_{\mathbf{q}}^{-k+1} \cdot k^{-a}$$

Theorem (Budd & Curien '17 for $a \neq 2$; Budd & Curien & ☺ '18 for $a = 2$).

	$3/2 < a < 2$	$a = 2$	$2 < a < 5/2$
$\lvert\overline{\mathrm{Ball}}_r(\mathfrak{M}_\infty^\dagger)\rvert$	e^{Cr}	$\mathrm{e}^{3\pi(r/2)^{1/2}}$	$r^{\frac{a-1/2}{a-2}}$
$\lvert\overline{\mathrm{Ball}}_r^{\mathrm{FPP}}(\mathfrak{M}_\infty^\dagger)\rvert$	∞		$r^{\frac{a-1/2}{a-2}}$

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Random walk

$(X_n)_{n\geq 0}$ simple random walk on $\mathfrak{M}_\infty$ and $(X_n^\dagger)_{n\geq 0}$ simple random walk on $\mathfrak{M}_\infty^\dagger$.
Recurrent or transient? diffusive?

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	$3/2 < a < 2$	$2 \leq a < 5/2$
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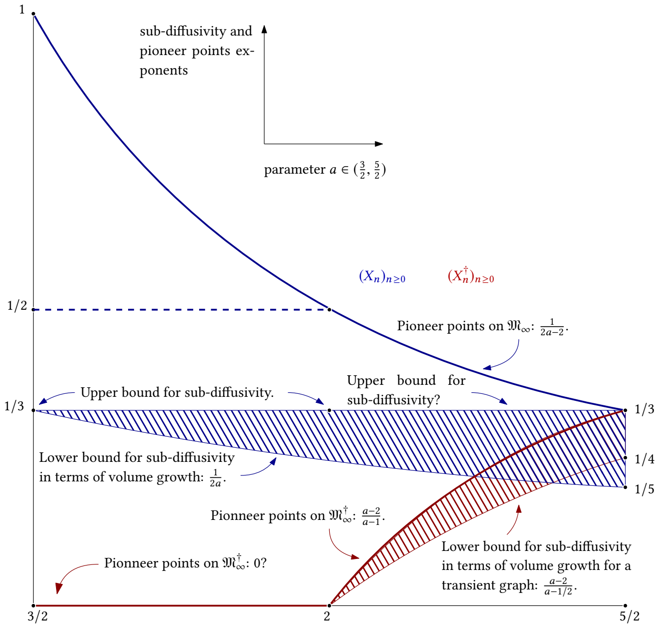
$(X_n^\dagger)_{n\geq 0}$	transient	transient?

THEOREM (Curien & ☺ in progress).

	$3/2 < a < 2$	$2 \leq a < 5/2$
$d(X_n, X_0) \lesssim n^?$	$1/3$	$1/(2a-2)$; $1/3$?
$d(X_n^\dagger, X_0^\dagger) \lesssim n^?$	0?	$(a-2)/(a-1)$

The big picture



Key tool: peeling maps

Peeling random planar maps

(Very) preliminary version

N. Curien

Gabriel Metsu (Dutch Baroque Era Painter, 1629-1667): Woman Peeling an Apple

Key tool: peeling maps

Y. Watabiki / Nuclear Physics B441 (1995) 119–163

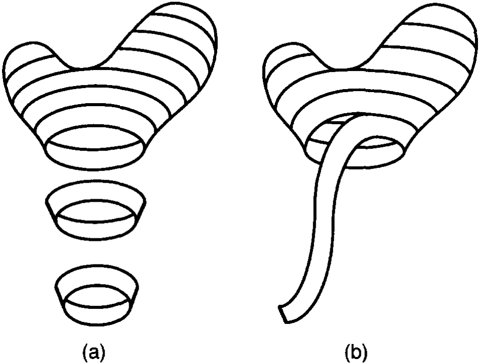
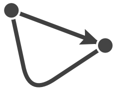
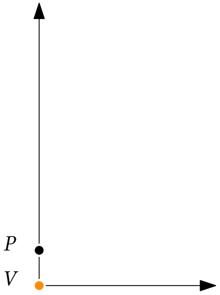


Fig. 4. Decomposition of a surface by (a) slicing and (b) peeling.

Key tool: peeling maps as Tim does





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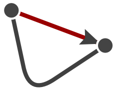

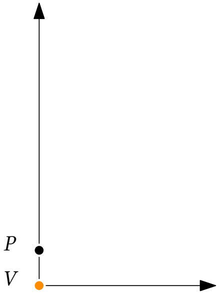

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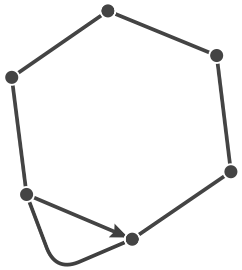

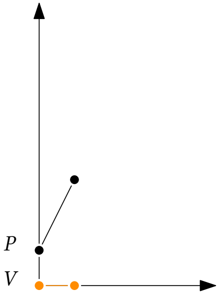

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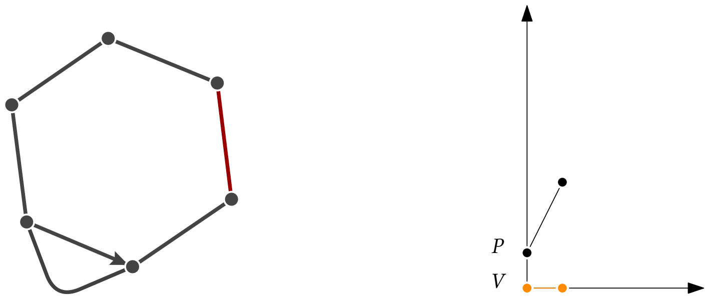

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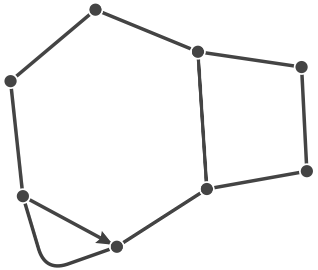

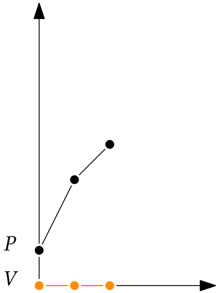

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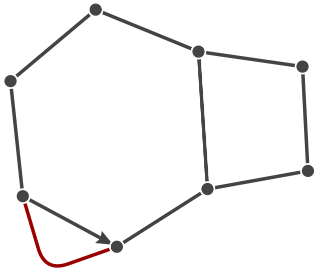

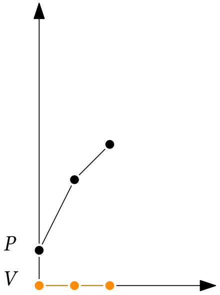

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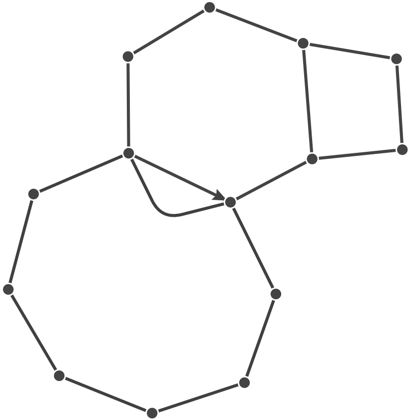

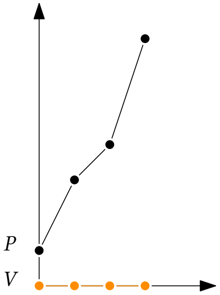

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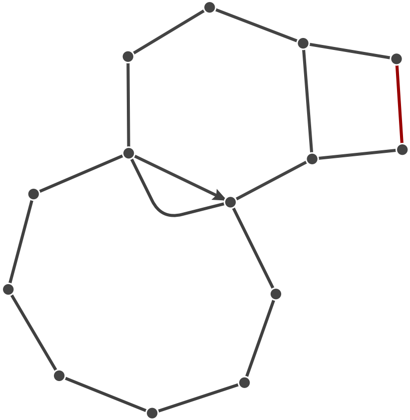

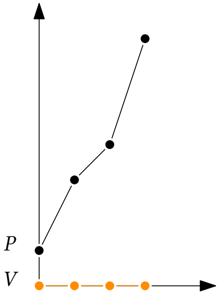

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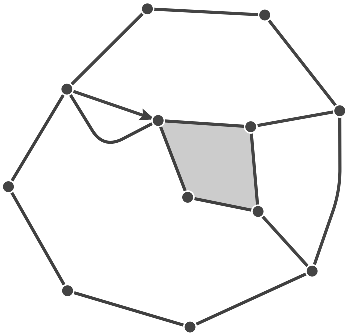

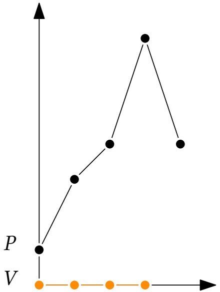

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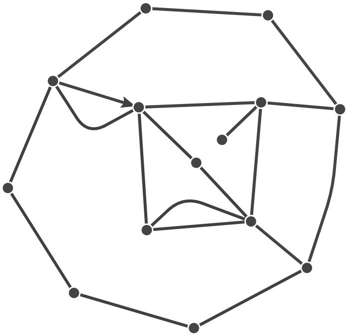

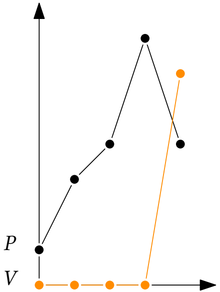

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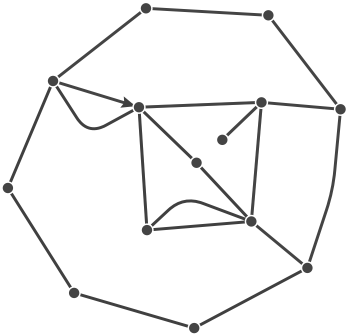
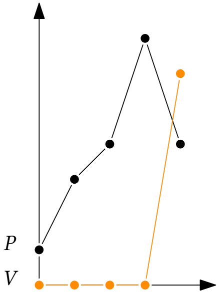




Budd '16: the pair $(P_n, V_n)_{n\geq 0}$ is a Markov chain started from $(1, 0)$. Precisely, from combinatorial considerations:

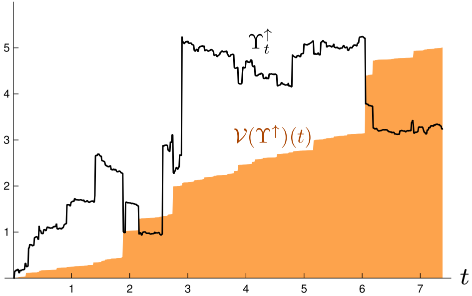
- $(P_n)_{n\geq 0}$ is some $(a-1)$-stable random walk *conditioned to stay positive.*
- Given $(P_n)_{n\geq 0}$, the increments $(V_{n+1} - V_n)_{n\geq 0}$ are independent and if $P_{n+1} - P_n + 1 = -\ell$, then $V_{n+1} - V_n$ has the law of $|B^{(\ell)}|$, the volume of a $\mathbf{q}$-Boltzmann map conditioned to have a root-face of degree 2ℓ.
- Budd & Curien '17: $\ell^{-(a-1/2)}|B^{(\ell)}| \to \xi$ in distribution.

Growth in any peeling process

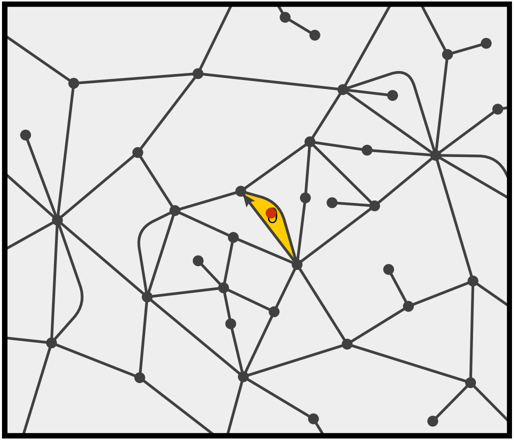
Theorem (Budd & Curien '17 for $a \neq 2$; Budd & Curien & ☺ '18 for $a = 2$). For every $a \in (\frac{3}{2}, \frac{5}{2})$, it holds that

$$\left(n^{-\frac{1}{a-1}} P_{[nt]}, n^{-\frac{a-1/2}{a-1}} V_{[nt]}\right)_{t\geq 0} \xrightarrow[n\to\infty]{(d)} \left(\Upsilon^{\uparrow}(t), \mathcal{V}(\Upsilon^{\uparrow})(t)\right)_{t\geq 0}.$$

$\Upsilon^{\uparrow}$ is a $(a-1)$-stable Lévy process *conditioned to stay positive* and $\mathcal{V}(\Upsilon^{\uparrow})$ is analogous to the discrete setting.



Peeling by layer



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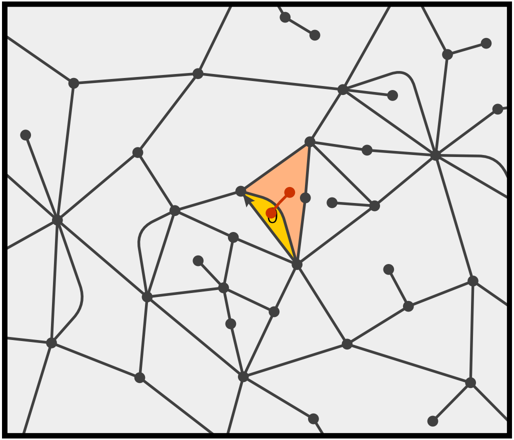

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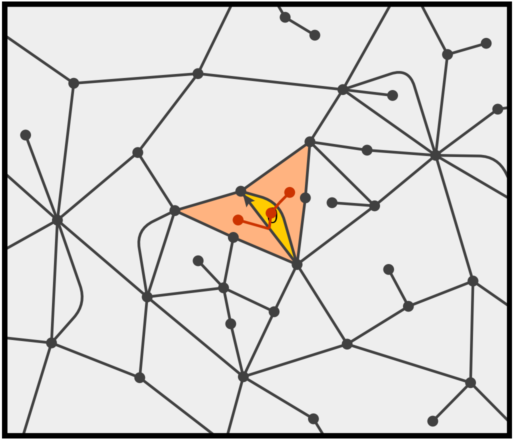

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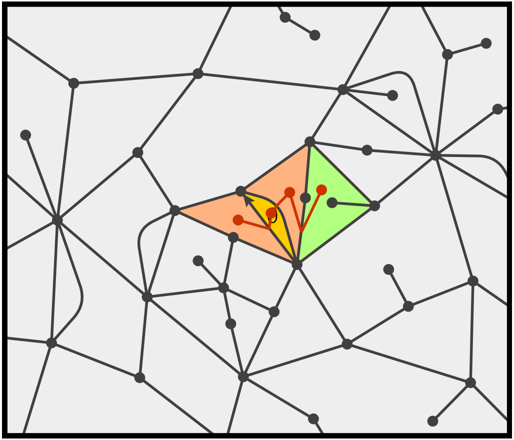

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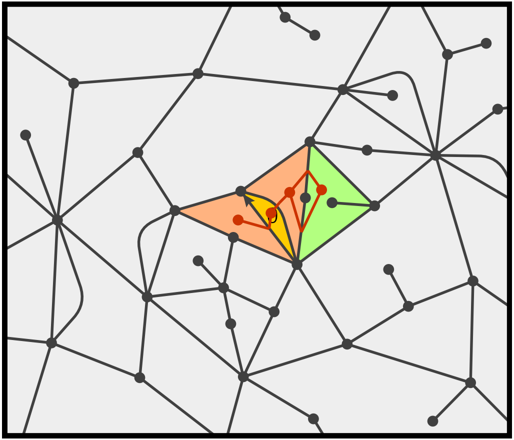

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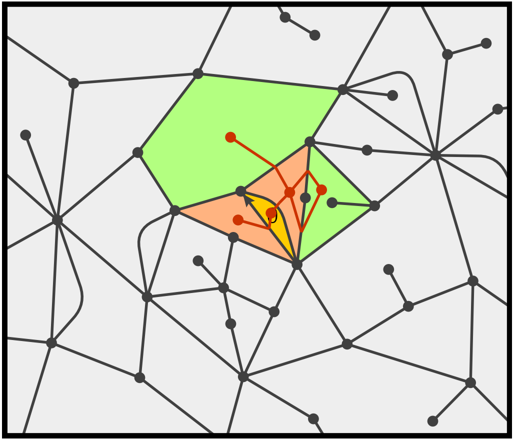

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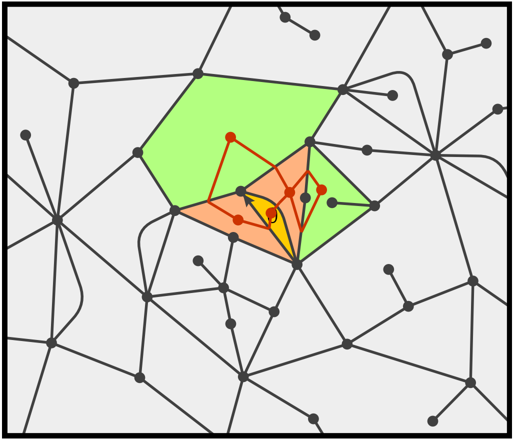

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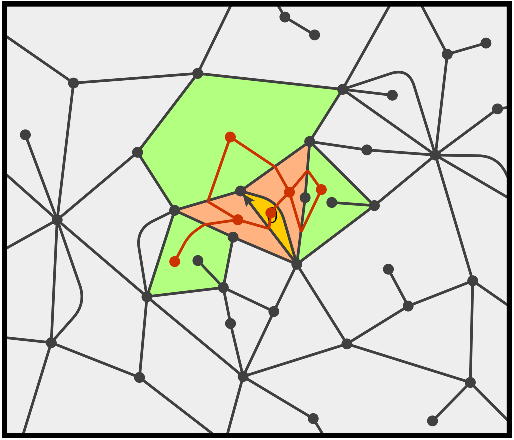

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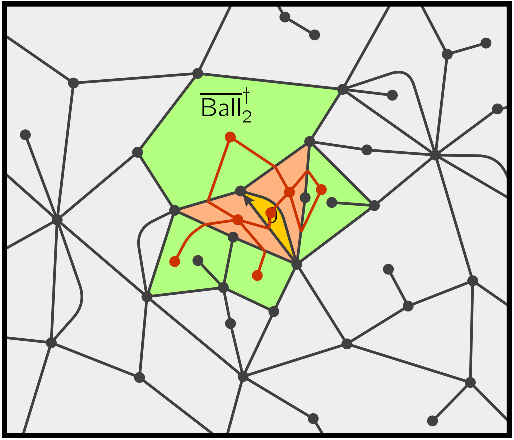

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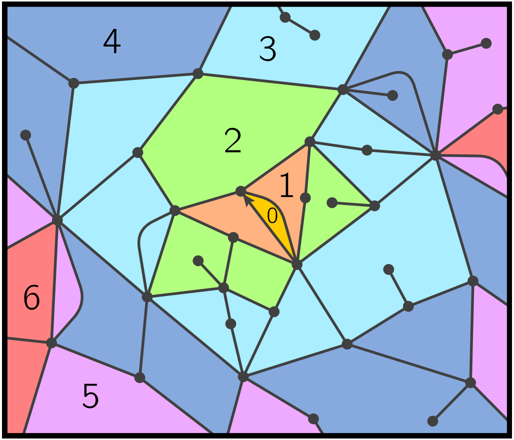


Volume growth in the dual graph distance

Can describe the dual graph distance via a peeling *by layer*.

$$H_n = \text{ distance reached after } n \text{ steps.}$$

$$\theta_r = \inf\{n : H_n \geq r\} = \text{ number of steps to reveal } \overline{\mathrm{Ball}}_r(\mathfrak{M}_\infty^\dagger).$$

Proposition.

$$\frac{H_n}{(\log n)^2} \xrightarrow[n\to\infty]{(\mathbf{P})} \frac{1}{2\pi^2}, \quad \text{equiv.} \quad \frac{\log \theta_r}{\sqrt{r}} \xrightarrow[n\to\infty]{(\mathbf{P})} \pi\sqrt{2}.$$

Heuristic: it takes $\approx \frac{2\ell}{\mathrm{p_q}\log \ell}$ steps to complete a layer of perimeter 2ℓ. Since $n^{-1}P_n \to \Upsilon_1^\uparrow$, we get

$$H_n \approx \sum_{k=0}^{n-1} \frac{\mathrm{p_q}\log P_k}{2P_k} \approx \sum_{k=1}^{n} \frac{\mathrm{p_q}\log k}{2\Upsilon_1^\uparrow k} \approx \frac{1}{\pi^2}\frac{(\log n)^2}{2}.$$

Consequently: since $n^{-3/2}V_n \to \mathcal{V}(\Upsilon^\uparrow)(1)$, we get

$$\frac{\log|\overline{\mathrm{Ball}}_r(\mathfrak{M}_\infty^\dagger)|}{\sqrt{r}} = \frac{\log V_{\theta_r}}{\log \theta_r}\cdot\frac{\log\theta_r}{\sqrt{r}} \xrightarrow[r\to\infty]{(\mathbf{P})} \frac{3}{2}\cdot\pi\sqrt{2} = \frac{3\pi}{\sqrt{2}}.$$

Volume growth in FPP for $a = 2$

Key observation: FPP in $\mathfrak{M}_\infty^\dagger$ = uniform random peeling in continuous time.

$$T_n = n\text{-th jump-time of the process } (|\overline{\mathrm{Ball}}_t^{\mathrm{FPP}}(\mathfrak{M}_\infty^\dagger)|)_t.$$

$$U_r = \inf\{n : T_n \geq r\} = \text{ number of steps to reveal } \overline{\mathrm{Ball}}_r^{\mathrm{FPP}}(\mathfrak{M}_\infty^\dagger).$$

Proposition.

$$\frac{T_n}{\log n} \xrightarrow[n\to\infty]{(\mathbf{P})} \frac{1}{\pi^2 \mathrm{p}_{\mathbf{q}}}, \quad \text{equiv.} \quad \frac{\log U_r}{r} \xrightarrow[n\to\infty]{(\mathbf{P})} \pi^2 \mathrm{p}_{\mathbf{q}}.$$

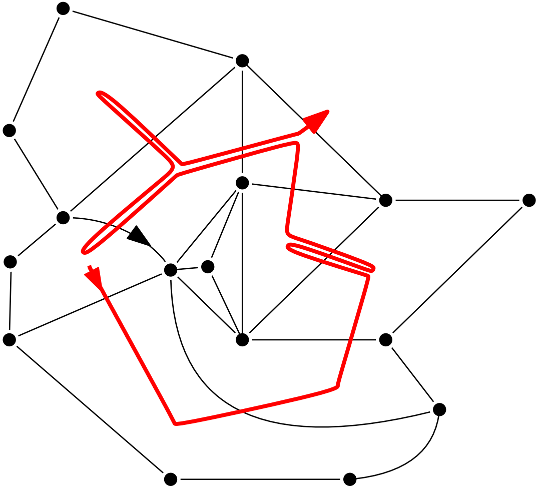
Heuristic: $T_{k+1} - T_k$ is the infimum of $2P_k$ i.i.d. Exp(1) so $T_{k+1} - T_k \approx \frac{1}{2P_k}$. Since, $k^{-1}P_k \to \Upsilon_1^\uparrow$, we get

$$T_n \approx \sum_{k<n} \frac{1}{2P_k} \approx \sum_{k<n} \frac{1}{2k\Upsilon_1^\uparrow} \approx \frac{\log n}{\pi^2 \mathrm{p}_{\mathbf{q}}}.$$

Consequently: since $n^{-3/2}V_n \to \mathcal{V}(\Upsilon^\uparrow)(1)$, we get

$$\frac{\log |\overline{\mathrm{Ball}}_r^{\mathrm{FPP}}(\mathfrak{M}_\infty^\dagger)|}{r} = \frac{\log U_r}{r} \cdot \frac{\log V_{U_r}}{\log U_r} \xrightarrow[r\to\infty]{(\mathbf{P})} \frac{3}{2} \cdot \pi^2 \mathrm{p}_{\mathbf{q}} = \frac{3\pi^2 \mathrm{p}_{\mathbf{q}}}{2}.$$

Pioneer points of the RW on the dual map



Pioneer points of the RW

$\max_{k \le n} d(o, P_n) \lesssim n^{\frac{1}{2(a-1)}}$? Idea: peeling along the walk.

For any peeling algorithm: if $D_n^{+/-}$ are the greatest/smallest distance from the origin to a vertex on the boundary of e_n the explored region after n peeling steps, then

$$\overline{\mathrm{Ball}}_{D_n^-}(\mathfrak{M}_\infty) \subset e_n \subset \overline{\mathrm{Ball}}_{D_n^+}(\mathfrak{M}_\infty).$$

Taking the volume:

$$(D_n^-)^{2a-1} \lesssim n^{\frac{a-1/2}{a-1}} \lesssim (D_n^+)^{2a-1}, \qquad \text{i.e.} \qquad D_n^- \lesssim n^{\frac{1}{2(a-1)}} \lesssim D_n^+.$$

Furthermore,

$$D_n^+ - D_n^- \le \text{aper}\, \mathfrak{M}_\infty^{(\partial e_n)} \lesssim |\partial e_n|^{1/2} \approx n^{\frac{1}{2(a-1)}} \qquad \text{so} \qquad D_n^- \approx n^{\frac{1}{2(a-1)}}.$$

Application with the peeling along the walk:

- Greatest distance to the origin of a pioneer point after n *peeling steps* $\approx n^{\frac{1}{2(a-1)}}$.
- Peeling only when the walk sits a pioneer point; logarithmic vertex-degrees $\Longrightarrow \approx n$ peeling steps when reaching the n-th pioneer point.

Sub-diffusivity of the RW

$$N_R = \#\text{ steps to exit } \mathrm{Ball}_R(\mathfrak{M}_\infty) \gtrsim R^\beta? \quad (\beta \geq 2)$$

General strategy: find a subset of edges $\mathcal{C}_R \subset \mathrm{Ball}_R(\mathfrak{M}_\infty) \setminus \mathrm{Ball}_{R/2}(\mathfrak{M}_\infty)$ such that

- $|\mathcal{C}_R| \gtrsim R^\gamma$;
- The walk has to traverse $\mathcal{C}_R$ to exit $\mathrm{Ball}_R(\mathfrak{M}_\infty)$;
- The walk flashed on $\mathcal{C}_R$ is diffusive;
- $\mathcal{C}_R$ defined in an 'ergodic' way, so $\mathbf{P}(k\text{-th step} \in \mathcal{C}_R) = \mathbf{P}(\text{root-edge} \in \mathcal{C}_R)$;
- $\mathbf{P}(\text{root-edge} \in \mathcal{C}_R) \lesssim R^{-\delta}$;

Then after N steps, the number of steps in $\mathcal{C}_R$ is $\lesssim N \cdot R^{-\delta}$; but when exiting $\mathrm{Ball}_R(\mathfrak{M}_\infty)$, the walk has made $\gtrsim R^{2\gamma}$ steps in $\mathcal{C}_R$ so

$$N_R \gtrsim R^{2\gamma+\delta}.$$

In our cases, $2\gamma + \delta = 3$.

Further questions

Discrete infinite maps:

- Exact value for sub-diffusive exponent? both on $\mathfrak{M}_\infty$ and $\mathfrak{M}_\infty^\dagger$.
- Transience of $\mathfrak{M}_\infty^\dagger$ for $a \geq 2$?

Scaling limits:

- $n^{-1/(2a-1)}\mathfrak{M}_\infty \to \mathfrak{M}$ *along subsequences* [Le Gall & Miermont '11, ☺ '18]. Uniqueness of the limit? 'stable map'
- $n^{-(a-1/2)/(a-2)}\mathfrak{M}_\infty^\dagger \to \mathfrak{M}^\dagger$? 'stable sphere'