Counting Injective Walks on Triangulations

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Abstract

In recent work (Barish & Suyama; *in preparation*) we extended Avi Wigderson's 1982 NP-completeness proof [7] for deciding the existence of Hamiltonian cycles on planar 3-vertex-connected triangulations, and obtained many-one counting ("weakly parsimonious") reductions from $\#SAT$ to the problem of counting Hamiltonian paths, simple cycles, and simple paths on this class of graphs. Here, we discuss some of the more interesting challenges that arose in constructing these reductions, and examine techniques for finding as well as proving generating functions, closed-form expressions, and (where possible) analytic expressions for the number of simple cycles and simple paths on infinite families of "sliced" planar 3-vertex-connected triangulations having bounded pathwidth.

Deciding the Existence of and Counting Hamiltonian Cycles & Paths on Planar 3-Vertex-Connected Triangulations

The Case of Hamiltonian Cycles (Almost Entirely Due to Wigderson [7])

Wigderson's proof [7] for the NP-completeness of the Hamiltonian cycle decision problem on planar 3-vertex-connected triangulations (i.e. maximal planar 3-vertex-connected graphs) actually provides a many-one counting reduction from counting Hamiltonian cycles on planar cubic (i.e. 3-valent) 3-vertex-connected graphs — which can easily be shown to be $\#P$-complete via minor modifications of an existing many-one counting reduction from $\#3SAT$ to counting Hamiltonian cycles on planar cubic 2-vertex-connected graphs [5] — to counting Hamiltonian cycles on planar 3-vertex-connected triangulations. However, we need to briefly note the existence of a minor error in Wigderson's proof wherein he mistakenly states that, beginning with an n vertex planar cubic 3-vertex-connected graph, the reduction construct will have $2^6 = 64^n$ Hamiltonian cycles per Hamiltonian cycle in the original graph; the actual amplification factor is $2^7 = 128^n$ due to there being another Hamiltonian cycle trajectory through the left-hand-side graph shown in Wigderson's "Figure 6" [7].

The Case of Hamiltonian Paths

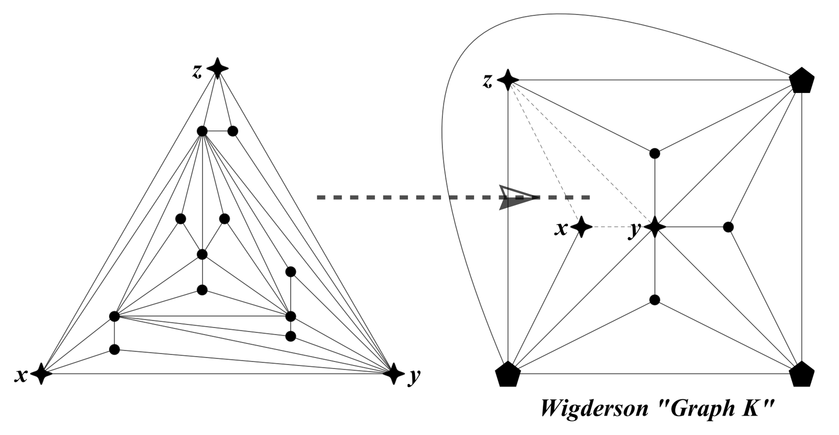


Figure 1: Scheme for the #(Hamiltonian Cycle) $\leq_m$ #(Hamiltonian Path) reduction.

To reduce the problem of counting Hamiltonian cycles to the problem of counting Hamiltonian paths on planar 3-vertex-connected triangulations, take any copy of Wigderson's "Graph K" (illustrated in "Figure 1" of Wigderson's proof [7] as well as in the above Figure 1) and perform the surgery illustrated in Figure 1 where we delete the dashed edges in our illustration of Wigderson's "Graph K" and identify the $\{x, y, z\}$ vertices of the resulting graph as the $\{x, y, z\}$ vertices of the left-hand-side gadget (found by scanning non-isometric planar graphs generated using Brinkmann and McKay's 'plantri' software [1]). A straightforward analysis — e.g. observing that all Hamiltonian cycles flowing through Wigderson's "Graph K" traverse the $\{x, z\}$ edge in our Figure 1 illustration of this same graph — shows that this will create exactly $2^8 = 256$ Hamiltonian paths (where both path ends are necessarily embedded internal to the left-hand-side gadget from Figure 1) per Hamiltonian cycle in the original graph, and eliminate all Hamiltonian cycles. Putting everything together, we have that there exists a many-one counting reduction from $\#3SAT$ to the problem of counting Hamiltonian paths on planar 3-vertex-connected triangulations.

Counting Simple Cycles and Simple Paths on Planar 3-Vertex-Connected Triangulations

In the context of this poster we will not fully reconstruct our many-one counting reduction from $\#SAT$ to counting simple cycles and simple paths on planar 3-vertex-connected triangulations, as the proof in (Barish & Suyama; *in preparation*) is > 10 pages in length and involves highly technical modifications of Wigderson's 1982 NP-completeness proof construction [7]. That said, we can elaborate on the "gadgeteering" part of this work, which arguably took on a life of its own. Here, we required an infinite family of graph gadgets — identifiable with selected faces in the triangulation given by a modified version of the Wigderson proof construction [7] — which: **(Criterion 1)** allowed us to ensure that the cardinality of the set of simple cycles or simple paths of length L, where L is the number of vertices in the graph prior to gadget identification surgeries, is larger than the cardinality of the set of simple cycles or simple paths of length $(L-1)$ by a factor that scales exponentially with a polynomially increasing gadget vertex count; **(Criterion 2)** allowed us to efficiently determine (in polynomial time) how many length L simple cycles (resp. length L simple paths) there are per Hamiltonian cycle (resp. Hamiltonian path) in the original graph, allowing us to recover the original Hamiltonian cycle (resp. Hamiltonian path) counts via integer division. This, in turn, required us to determine closed-form (and when possible, analytic) expressions for all possible manners in which simple cycles and simple paths can ingress and egress the aforementioned gadget, which we refer to as "path amplification factors". For our infinite family of graph gadgets, we chose the infinite family of "sliced" planar 3-vertex-connected triangulations shown in Figure 2. In consideration of the bounded pathwidth of the Figure 2 triangulation, we ask the reader to observe that Courcelle's theorem and its extensions only yields a guarantee for **(Criterion 2)**.

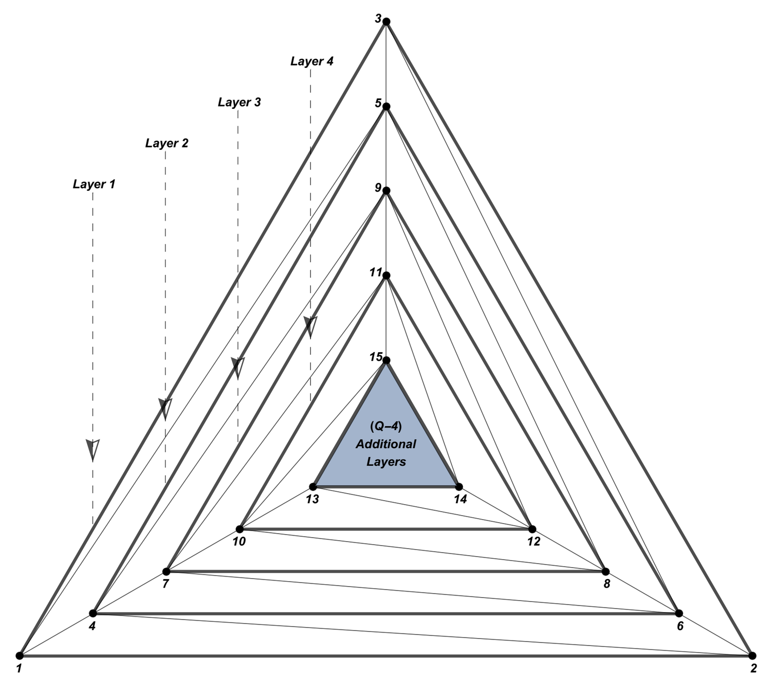
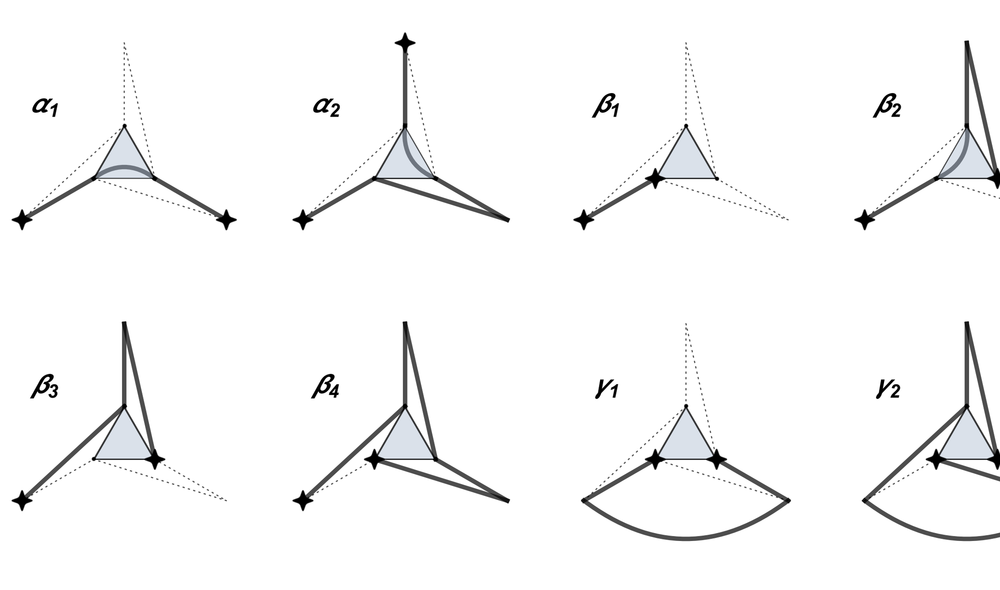


Figure 2: Illustration of an infinite family of "sliced" planar 3-vertex-connected triangulations having pathwidth and treewidth exactly 4 iff there are at least two layers (otherwise the pathwidth and treewidth will be exactly 2). Note that pathwidth is equal to the vertex separation number of a graph [4], and that the provided vertex linear ordering achieves the minimum vertex separation number.

Characterization of Path Amplification Factors for the Figure 2 Triangulation



Inner Layer (r+1) Traversal Type

Outer Layer (r) Traversal Type	α_1	α_2	β_1	β_2	β_3	β_4	γ_1	γ_2	STOP	END
α_1	9	2	0	0	0	0	0	0	6	2
α_2	3	0	0	0	0	0	0	0	2	1
β_1	18	4	10	12	8	2	0	0	13	5
β_2	3	0	2	2	2	0	0	0	2	1
β_3	3	0	4	2	1	0	0	0	3	2
β_4	0	0	2	0	0	0	0	0	1	1
γ_1	6	0	12	8	6	0	9	2	5	3
γ_2	0	0	4	0	0	0	3	0	1	1

Figure 3: **(Top)** Illustration of all possible manners in which a simple path can ingress and egress a layer of the "sliced" planar 3-vertex-connected triangulation shown in Figure 2 starting from an outer layer; "concave diamond" shaped vertices illustrate the position of path ends. **(Bottom)** State transition matrix where integer values correspond to the number of ways a path traversing an outer layer via one of the transition types $\{\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \beta_4, \gamma_1, \gamma_2\}$ (row labels) may ingress into (and possibly egress from) the next layer via one of the transition types $\{\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \beta_4, \gamma_1, \gamma_2\}$ (column labels); transitions to "STOP" correspond to instances where paths do not ingress all layers of the relevant instance of the Figure 2 triangulation, and transitions to "END" correspond to instances where paths ingress all layers of the relevant instance of the Figure 2 triangulation.

If one enumerates all paths in the Directed Acyclic Graph (DAG) associated with the state transition matrix from Figure 3(Bottom), assigns integer weights to the edges of the paths based on the transitions they correspond to in the matrix, and sums over the products of the edge weights for each path, then one can generate the path amplification factors for $\{\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \beta_4, \gamma_1, \gamma_2\}$ traversals out to ≈ 30 layers on a standard personal computer. After performing the aforementioned computation, we utilized the 'guessPRec[]' algorithm due to Hebisch and Rubey [2], running in the context of the FriCAS (version 1.2.4) fork of Axiom [3], to guess recurrence relations (more specifically, linear recurrence relation with polynomial coefficients) for each path amplification factor, then invoked the Mathematica (version 10.4.1) 'RSolve[]' algorithm [8] to find closed-form expressions. We also utilized Doron Zeilberger's 'guessHolo[]' algorithm [9] to find (holonomic) generating functions for each path amplification factor. In the below summary of our calculations, when listing explicit integer values for each path amplification factor we (**Bold**) the minimum number of terms necessary for the 'guessPRec[]' routine to find a recurrence, and highlight in (purple) the minimum number of additional terms required for the 'guessHolo[]' algorithm to yield a generating function.

$$\Psi_{paths(\alpha_1)}[q] = \left(\frac{1}{105\left(\sqrt{105}+9\right)}\right)\left(\left(29\sqrt{105}+285\right)\left(\frac{1}{2}\left(\sqrt{105}+9\right)\right)^q + \left(11\sqrt{105}+75\right)\left(\frac{1}{2}\left(9-\sqrt{105}\right)\right)^q - 75\left(\sqrt{105}+9\right)\right)$$

$$\mathcal{G}_{(\alpha_1)} = \sum_{n=0}^{\infty} a_n x^n = \left(\frac{2x^2+6x+2}{6x^3+3x^2-10x+1}\right)$$

$$q = (1\ to\ 8) \longrightarrow \{\mathbf{2}, \mathbf{26}, \mathbf{256}, \mathbf{2470}, \mathbf{23776}, \mathbf{228814}, 2201992, 21190822\}$$

$$\Psi_{paths(\alpha_2)}[q] = \left(\frac{1}{70\left(\sqrt{105}+9\right)}\right)\left(4\left(\sqrt{105}+20\right)\left(\frac{\sqrt{105}}{2}+\frac{9}{2}\right)^q - \left(29\sqrt{105}+305\right)\left(\frac{1}{2}\left(9-\sqrt{105}\right)\right)^q - 10\left(\sqrt{105}+9\right)\right)$$

$$\mathcal{G}_{(\alpha_2)} = \left(\frac{3x^2-2x+1}{6x^3+3x^2-10x+1}\right)$$

$$q = (1\ to\ 8) \longrightarrow \{\mathbf{1}, \mathbf{8}, \mathbf{80}, \mathbf{770}, \mathbf{7412}, \mathbf{71330}, 686444, 6605978\}$$

$$\Psi_{paths(\beta_1)}[q] = \ldots (closed\ form\ exp.\ exists;\ contains\ \leq 989\ polynomial\ kth\ root\ functions)$$

$$\mathcal{G}_{(\beta_1)} = \left(\frac{24x^5+64x^4-46x^3-137x^2+18x+5}{48x^7-24x^6-296x^5-86x^4+279x^3+101x^2-23x+1}\right)$$

$$q = (1\ to\ 16) \longrightarrow \{\mathbf{5}, \mathbf{133}, \mathbf{2417}, \mathbf{40717}, \mathbf{655761}, \mathbf{10308685}, \mathbf{159755201}, \mathbf{2454455165}, \mathbf{37509569681}, \mathbf{571329943277}, \mathbf{8684133268449}, \mathbf{131824157577949}, \mathbf{1999412791034545}, \mathbf{30309669542946125}, 30309669542946125, 459319499169145089\}$$

$$\Psi_{paths(\beta_2)}[q] = \ldots (closed\ form\ exp.\ exists;\ contains\ \leq 977\ polynomial\ kth\ root\ functions)$$

$$\mathcal{G}_{(\beta_2)} = \left(\frac{-36x^5-130x^4+35x^3+9x^2+x+1}{48x^7-24x^6-296x^5-86x^4+279x^3+101x^2-23x+1}\right)$$

$$q = (1\ to\ 16) \longrightarrow \{\mathbf{1}, \mathbf{24}, \mathbf{460}, \mathbf{7912}, \mathbf{128776}, \mathbf{2036720}, \mathbf{31677424}, \mathbf{487760648}, \mathbf{7464256776}, \mathbf{113789522528}, \mathbf{1730513701408}, \mathbf{26277912349816}, \mathbf{398649789148408}, \mathbf{6044068185071440}, 91601063613696144, 1387924790432306408\}$$

$$\Psi_{paths(\beta_3)}[q] = \ldots (closed\ form\ exp.\ exists;\ contains\ \leq 965\ polynomial\ kth\ root\ functions)$$

$$\mathcal{G}_{(\beta_3)} = \left(\frac{68x^4-18x^3+137x^2-13x+2}{48x^7-24x^6-296x^5-86x^4+279x^3+101x^2-23x+1}\right)$$

$$q = (1\ to\ 16) \longrightarrow \{\mathbf{2}, \mathbf{33}, \mathbf{694}, \mathbf{12053}, \mathbf{198158}, \mathbf{3150085}, \mathbf{49144710}, \mathbf{758126341}, \mathbf{11615040766}, \mathbf{177193621125}, \mathbf{2695979966838}, \mathbf{40950198919877}, \mathbf{621347905384430}, \mathbf{9421545861756101}, 142798856837832294, 2163764931932764293\}$$

$$\Psi_{paths(\beta_4)}[q] = \ldots (closed\ form\ exp.\ exists;\ contains\ \leq 977\ polynomial\ kth\ root\ functions)$$

$$\mathcal{G}_{(\beta_4)} = \left(\frac{104x^5+180x^4+84x^3+115x^2-12x+1}{48x^7-24x^6-296x^5-86x^4+279x^3+101x^2-23x+1}\right)$$

$$q = (1\ to\ 16) \longrightarrow \{\mathbf{1}, \mathbf{11}, \mathbf{267}, \mathbf{4835}, \mathbf{81435}, \mathbf{1311523}, \mathbf{20617371}, \mathbf{319510403}, \mathbf{4908910331}, \mathbf{75019139363}, \mathbf{1142659886555}, \mathbf{17368266536899}, \mathbf{263648315155899}, \mathbf{3998825582069091}, 60619339085892251, 918638998338290179\}$$

$$\Psi_{paths(\gamma_1)}[q] = \ldots (closed\ form\ exp.\ exists;\ contains\ \leq 20592\ polynomial\ kth\ root\ functions)$$

$$\mathcal{G}_{(\gamma_1)} = \left(\frac{-96x^7-464x^6-208x^5+632x^4+56x^3-215x^2-30x-3}{288x^9+288x^8-2040x^7-3156x^6+1196x^5+3203x^4+492x^3-302x^2+32x-1}\right)$$

$$q = (1\ to\ 20) \longrightarrow \{\mathbf{3}, \mathbf{126}, \mathbf{3341}, \mathbf{70280}, \mathbf{1310947}, \mathbf{22776890}, \mathbf{378375161}, \mathbf{6103069292}, \mathbf{96507366351}, \mathbf{1505563065374}, \mathbf{23270400625317}, \mathbf{357380529152848}, \mathbf{5464425118434555}, \mathbf{83299436167233410}, \mathbf{1267175626578743633}, \mathbf{19249262154903884852}, \mathbf{292126169910413863463}, \mathbf{4430379317943310914790}, 67161093852565692523005, 1017803522770373586836760\}$$

$$\Psi_{paths(\gamma_2)}[q] = \ldots (closed\ form\ exp.\ exists;\ contains\ \leq 20592\ polynomial\ kth\ root\ functions)$$

$$\mathcal{G}_{(\gamma_2)} = \left(\frac{432x^7+1944x^6+1316x^5-724x^4+892x^3-253x^2+2x-1}{288x^9+288x^8-2040x^7-3156x^6+1196x^5+3203x^4+492x^3-302x^2+32x-1}\right)$$

$$q = (1\ to\ 20) \longrightarrow \{\mathbf{1}, \mathbf{30}, \mathbf{911}, \mathbf{19692}, \mathbf{373709}, \mathbf{6555886}, \mathbf{109565411}, \mathbf{1774146288}, \mathbf{28127028537}, \mathbf{439560377778}, \mathbf{6802008969231}, \mathbf{104547734949748}, \mathbf{1599438217770341}, \mathbf{24390926519441846}, \mathbf{371136986673484731}, \mathbf{5638804876412811256}, \mathbf{85584406387962049937}, \mathbf{1298074700875871029690}, 19678855995668252304743, 298237123445626712249980\}$$

Inductive Proofs of Closed-Form Expressions for Path Amplification Factors

We can prove the analytic or closed-form expressions for the path amplification factors by establishing the below equalities using a computer algebra system; we utilized Mathematica 10.4.1 [8].

$$\Psi_{paths(\alpha_1)}[q \to (q+1)] = 9\left(\Psi_{paths(\alpha_1)}[q]\right) + 2\left(\Psi_{paths(\alpha_2)}[q]\right) + 6$$

$$\Psi_{paths(\alpha_2)}[q \to (q+1)] = 3\left(\Psi_{paths(\alpha_1)}[q]\right) + 2$$

$$\Psi_{paths(\beta_1)}[q \to (q+1)] = 18\left(\Psi_{paths(\alpha_1)}[q]\right) + 4\left(\Psi_{paths(\alpha_2)}[q]\right) + 10\left(\Psi_{paths(\beta_1)}[q]\right) + 12\left(\Psi_{paths(\beta_2)}[q]\right) + 8\left(\Psi_{paths(\beta_3)}[q]\right) + 2\left(\Psi_{paths(\beta_4)}[q]\right) + 13$$

$$\Psi_{paths(\beta_2)}[q \to (q+1)] = 3\left(\Psi_{paths(\alpha_1)}[q]\right) + 2\left(\Psi_{paths(\beta_1)}[q]\right) + 2\left(\Psi_{paths(\beta_2)}[q]\right) + 2\left(\Psi_{paths(\beta_3)}[q]\right) + 2$$

$$\Psi_{paths(\beta_3)}[q \to (q+1)] = 3\left(\Psi_{paths(\alpha_1)}[q]\right) + 4\left(\Psi_{paths(\beta_1)}[q]\right) + 2\left(\Psi_{paths(\beta_2)}[q]\right) + \left(\Psi_{paths(\beta_3)}[q]\right) + 3$$

$$\Psi_{paths(\beta_4)}[q \to (q+1)] = 2\left(\Psi_{paths(\beta_1)}[q]\right) + 1$$

$$\Psi_{paths(\gamma_1)}[q \to (q+1)] = 6\left(\Psi_{paths(\alpha_1)}[q]\right) + 12\left(\Psi_{paths(\beta_1)}[q]\right) + 8\left(\Psi_{paths(\beta_2)}[q]\right) + 6\left(\Psi_{paths(\beta_3)}[q]\right) + 9\left(\Psi_{paths(\gamma_1)}[q]\right) + 2\left(\Psi_{paths(\gamma_2)}[q]\right) + 5$$

$$\Psi_{paths(\gamma_2)}[q \to (q+1)] = 4\left(\Psi_{paths(\beta_1)}[q]\right) + 3\left(\Psi_{paths(\gamma_1)}[q]\right) + 1$$

Inductive Proofs of Generating Functions for Path Amplification Factors

We can prove the generating functions for our path amplification factors by establishing the below equalities with a computer algebra system. We again utilized Mathematica 10.4.1 [8], though unlike in the case of proving our closed-form expressions, with some "hand-holding" almost any computer algebra system should be up to the task (e.g. SageMath [6] or even Maxima). Note that coefficients of the power series expansion about $x = 0$ for the (red) "adjustment" terms are equal to zero up to all orders.

$$\mathcal{G}_{(\alpha_1)} \times x^{-1} = 9\mathcal{G}_{(\alpha_1)} + 2\mathcal{G}_{(\alpha_2)} + \left(\tfrac{-6}{x-1}\right) + \left(\tfrac{2}{x}\right)$$

$$\mathcal{G}_{(\alpha_2)} \times x^{-1} = 3\mathcal{G}_{(\alpha_1)} + \left(\tfrac{-2}{x-1}\right) + \left(\tfrac{1}{x}\right)$$

$$\mathcal{G}_{(\beta_1)} \times x^{-1} = 18\mathcal{G}_{(\alpha_1)} + 4\mathcal{G}_{(\alpha_2)} + 10\mathcal{G}_{(\beta_1)} + 12\mathcal{G}_{(\beta_2)} + 8\mathcal{G}_{(\beta_3)} + 2\mathcal{G}_{(\beta_4)} + \left(\tfrac{-13}{x-1}\right) + \left(\tfrac{5}{x}\right)$$

$$\mathcal{G}_{(\beta_2)} \times x^{-1} = 3\mathcal{G}_{(\alpha_1)} + 2\mathcal{G}_{(\beta_1)} + 2\mathcal{G}_{(\beta_2)} + 2\mathcal{G}_{(\beta_3)} + \left(\tfrac{-2}{x-1}\right) + \left(\tfrac{1}{x}\right)$$

$$\mathcal{G}_{(\beta_3)} \times x^{-1} = 3\mathcal{G}_{(\alpha_1)} + 4\mathcal{G}_{(\beta_1)} + 2\mathcal{G}_{(\beta_2)} + \mathcal{G}_{(\beta_3)} + \left(\tfrac{-3}{x-1}\right) + \left(\tfrac{2}{x}\right)$$

$$\mathcal{G}_{(\beta_4)} \times x^{-1} = 2\mathcal{G}_{(\beta_1)} + \left(\tfrac{-1}{x-1}\right) + \left(\tfrac{1}{x}\right)$$

$$\mathcal{G}_{(\gamma_1)} \times x^{-1} = 6\mathcal{G}_{(\alpha_1)} + 12\mathcal{G}_{(\beta_1)} + 8\mathcal{G}_{(\beta_2)} + 6\mathcal{G}_{(\beta_3)} + 9\mathcal{G}_{(\gamma_1)} + 2\mathcal{G}_{(\gamma_2)} + \left(\tfrac{-5}{x-1}\right) + \left(\tfrac{3}{x}\right)$$

$$\mathcal{G}_{(\gamma_2)} \times x^{-1} = 4\mathcal{G}_{(\beta_1)} + 3\mathcal{G}_{(\gamma_1)} + \left(\tfrac{-1}{x-1}\right) + \left(\tfrac{1}{x}\right)$$

Closed-Form Expressions of Simple Cycle and Simple Path Counts

Letting q be the number of layers of the "sliced" planar 3-vertex-connected triangulation shown in Figure 2, we can write down analytic expression and closed-form expressions, respectively, for the number of simple cycles (Φ_{cycles}) and the number of simple paths (Φ_{paths}) in the triangulation.

$$\Phi_{cycles} = 21\sum_{r=1}^{(q-1)} \Psi_{paths(\alpha_1)}[r] + 9\sum_{r=1}^{(q-1)} \Psi_{paths(\alpha_2)}[r] + 10(q-1) + q$$

$$\Longrightarrow \Phi_{cycles} = \left(\frac{1}{490\left(\sqrt{105}+9\right)}\right)\left(4\left(\left(101\sqrt{105}+840\right)\left(\frac{1}{2}\left(\sqrt{105}+9\right)\right)^q - 70\left(\sqrt{105}+9\right)\right) + \left(611\sqrt{105}+5775\right)\left(\frac{1}{2}\left(9-\sqrt{105}\right)\right)^q - 2590\left(\sqrt{105}+9\right)q\right)$$

$$q = (1\ to\ 8) \longrightarrow \{1, 63, 692, 6799, 65610, 631625, 6078700, 58498539\}$$

$$\Phi_{paths} = 27\sum_{r=1}^{(q-1)} \Psi_{paths(\alpha_1)}[r] + 6\sum_{r=1}^{(q-1)} \Psi_{paths(\alpha_2)}[r] + 30\sum_{r=1}^{(q-1)} \Psi_{paths(\beta_1)}[r] + 36\sum_{r=1}^{(q-1)} \Psi_{paths(\beta_2)}[r] + 24\sum_{r=1}^{(q-1)} \Psi_{paths(\beta_3)}[r] + 6\sum_{r=1}^{(q-1)} \Psi_{paths(\beta_4)}[r] + 21\sum_{r=1}^{(q-1)} \Psi_{paths(\gamma_1)}[r] + 9\sum_{r=1}^{(q-1)} \Psi_{paths(\gamma_2)}[r] + 12(q-1) + 6q$$

$$q = (1\ to\ 8) \longrightarrow \{6, 396, 9792, 202890, 3751950, 64884828, 1074862116, 17306622222\}$$

Primary Findings (Barish & Suyama; *in preparation*)

(Finding 1): Deciding the existence of a Hamiltonian path on a planar 3-vertex-connected triangulation is NP-complete.

(Finding 2): There exists a many-one counting reduction from $\#SAT$ to the problem of counting Hamiltonian paths, simple cycles, and simple paths on planar 3-vertex-connected triangulations.
$\Longrightarrow$ Counting these objects is $\#P$-complete under many-one counting ("weakly parsimonious") reductions.
$\Longrightarrow$ No Fully Polynomial-time Randomized Approximation Scheme (FPRAS) unless $NP = RP$.

(Finding 3): Existence of an analytic expression for the number of simple cycles in the family of "sliced" planar 3-vertex-connected triangulations illustrated in Figure 2.

(Finding 4): Existence of a closed-form expression for the number of simple paths in the family of "sliced" planar 3-vertex-connected triangulations illustrated in Figure 2 (the determination of which is likely at the edge of what's possible with a modern computer algebra system like Mathematica or Maple).

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