Natural parametrization for the scaling limit of loop-erased random walk in three dimensions

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joint work with Xinyi Li (University of Chicago)

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- Discretization: Consider a discrete random path and take its scaling limit.
- Our choice of the discretization is **loop-erased random walk (LERW)**.
- LERW is the **only** model with self-repulsion that we can analyze rigorously for $d = 3$.

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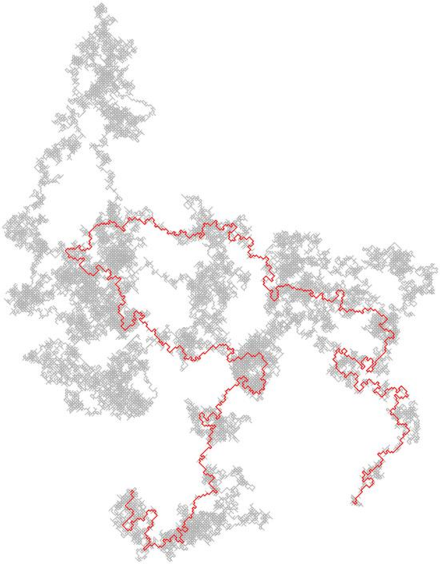
- **Interest**: Scaling limit of 3D LERW.
- **No similar procedure to erase loops from Brownian motion!**



Picture credit: Fredrik Viklund.

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- Another interpretation: LERW = **Laplacian random walk**. Namely, for $x \sim \gamma^{(n)}(k)$.

$$P\left(\gamma^{(n)}(k+1) = x \;\middle|\; \gamma^{(n)}[0, k]\right) = \frac{f(x)}{\sum_{y\sim\gamma^{(n)}(k)} f(y)}.$$

Here f is discrete harmonic on $\left(\mathbb{D} \cap \frac{1}{n}\mathbb{Z}^3\right) \setminus \gamma^{(n)}[0, k]$ with $f \equiv 1$ on $\partial\left(\mathbb{D} \cap \frac{1}{n}\mathbb{Z}^3\right)$ and $f \equiv 0$ on $\gamma^{(n)}[0, k]$.

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- **Question**: What is the scaling limit of $\gamma^{(n)}$ as $n \to \infty$? What is the topology for the limit?

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- (Kozma '07): As $n \to \infty$, $\gamma^{(2^n)}$ converges weakly to some random compact set $\mathcal{K}$ w,r,t, the Hausdorff distance d_{Haus}.
- Kozma shows that $\gamma^{(2^n)}$ is a Cauchy sequence w.r.t. the Prokhorov metric. No nice tool like SLE to describe $\mathcal{K}$!

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- (S. '16) W.p.1, $\dim(\mathcal{K}) = \beta$.

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Theorem (Li - S. '18)

There exist universal $c_0 > 0$, $\delta > 0$ *such that*

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- From this theorem, $M_{2^n}/2^{\beta n}$ is tight.

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- Let η_n be the time rescaled $\gamma^{(2^n)}$ defined by

$$\eta_n(t) = \gamma^{(2^n)}(2^{\beta n} t) \quad \text{for} \quad 0 \le t \le M_{2^n}/2^{\beta n},$$

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$$\mu_n = 2^{-\beta n} \sum_{x \in \gamma^{(2^n)} \cap 2^{-n}\mathbb{Z}^3} \delta_x$$

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Theorem (Li - S. '18 work in progress)

As $n \to \infty$, $(\gamma^{(2^n)}, \mu_n)$ converges weakly to some $(\mathcal{K}, \mu)$ w.r.t. the product topology of $\mathcal{H}(\overline{\mathbb{D}})$ and $\mathcal{M}(\overline{\mathbb{D}})$, where $\mathcal{K}$ is Kozma's scaling limit. Furthermore, the measure μ is a measurable function of $\mathcal{K}$.

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W.p.1, supp$(\mu) = \mathcal{K}$. *Moreover, it follows that w.p.1, for each* $x \in \mathcal{K}$

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- Take two continuous curves $\lambda_i : [0, t_i] \to \overline{\mathbb{D}}$ $(i = 1, 2)$. Let
$$\rho(\lambda_1, \lambda_2) = |t_1 - t_2| + \max_{0 \le s \le 1} \left|\lambda_1(st_1) - \lambda_2(st_2)\right|$$
be the supremum distance of them.

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  - (Lawler 1993) For $d = 4$, $\exists c_4 > 0$ s.t. $\gamma^{(n)}\left(c_4 n^2 (\log n)^{-\frac{1}{3}} t\right) \to$ BM.

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  - (Lawler 1993) For $d = 4$, $\exists c_4 > 0$ s.t. $\gamma^{(n)}\left(c_4 n^2 (\log n)^{-\frac{1}{3}} t\right) \to$ BM.
  - (Lawler-Viklund 2017) For $d = 2$, $\exists c_2 > 0$ s.t. $\gamma^{(n)}\left(c_2 n^{\frac{5}{4}} t\right) \to \gamma$ where γ is SLE_2 parametrized by 5/4-dimensional Minkowski content.

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Theorem (Li - S. '18 work in progress)

There exists a random continuous curve $\lambda : [0,\infty) \to \mathbb{R}^3$ such that as $n \to \infty$, $\lambda_n\big(2^{\beta n}\,\cdot\,\big)$ converges weakly to λ with respect to the metric χ.

What are η and λ?

A big challenging problem is to find a “nice” way to describe η and λ.

Thank you for your attention!