Random volumes from matrices

Masafumi Fukuma
(Dept of Phys, Kyoto Univ)

Physics and Mathematics of Discrete Geometries
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based on work with

Sotaro Sugishita (U Kentucky) & **Naoya Umeda** (PwC)

[arXiv:1503.08812] "Random volumes from matrices"

[arXiv:1504.03532] "Matter fields in triangle-hinge models"

[arXiv:1603.05199] "Triangle-hinge models for unoriented membranes"

[arXiv:181*.***]   "Critical behavior of triangle-hinge models"**

0. Introduction

String / M theory

String theory : strong candidate for the unified theory including 4D QG

[worldsheet picture]

2D QG + matters

(intrinsic metric) (target space coords)

$h_{ab}(\xi)$ $X^{\mu}(\xi)$ $\left(\xi = (\xi^a) = (\tau, \sigma)\right)$

M theory : an approach to string theory:

⇔ (super)membranes in 11D spacetime

[worldvolume picture]

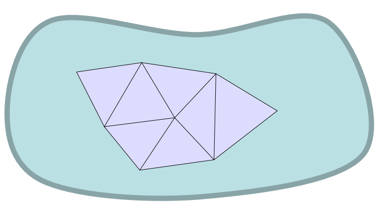
3D QG + matters

(intrinsic metric) (target space coords)

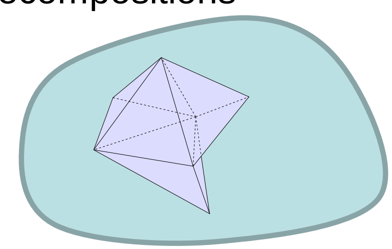
$h_{ab}(\xi)$ $X^{\mu}(\xi)$ $\left(\xi = (\xi^a) = (\tau, \sigma^1, \sigma^2)\right)$

Strings and membranes (1/3)

	string	M
fundamental constituents	**string**	**membrane**
trajectory	**worldsheet**	**worldvolume**
system	**<u>2D QG</u> + matters**	**<u>3D QG</u> + matters**
	$\Updownarrow$	$\Updownarrow$
	random surfaces	**random volumes**
free energy $\log Z$	$\sum_{\text{conn. triangular decompositions}} e^{-S}$	$\sum_{\text{conn. tetrahedral decompositions}} e^{-S}$



$M = S^2$



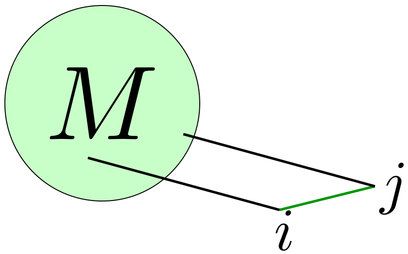
$M = S^3$

Strings and membranes (2/3)

random surfaces

generating model **matrix model**

$$M = (M_{ij})$$



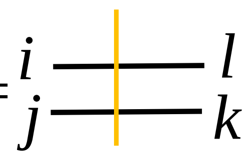
$$S(M) = \frac{1}{2}\mathrm{tr}M^2 - \frac{\lambda}{3}\mathrm{tr}M^3$$

$$= \frac{1}{2}M_{ij}M_{ji} - \frac{\lambda}{3}M_{ij}M_{jk}M_{ki}$$

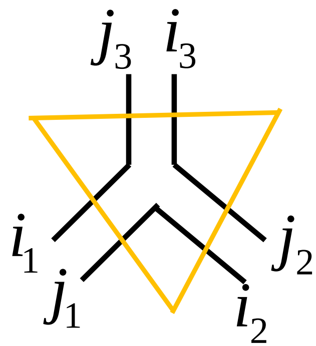
Strings and membranes (3/3)

random surfaces

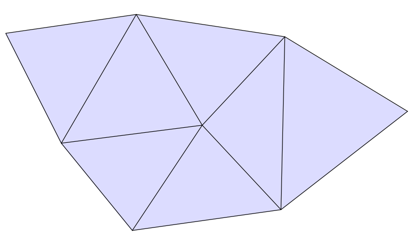
$$\left\langle M_{ij} M_{kl} \right\rangle_0 =$$



$$= \delta_{il} \delta_{jk}$$

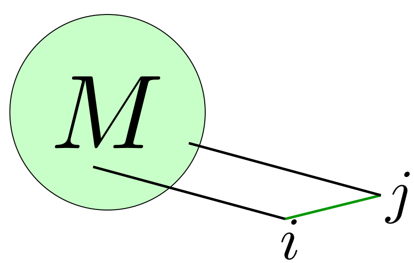


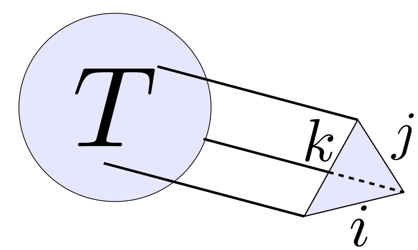
$$= \lambda \, \delta_{j_1 i_2} \delta_{j_2 i_3} \delta_{j_3 i_1}$$



Strings and membranes (2′/3)

	random surfaces	random volumes
generating model	**matrix model**	**tensor model (?)**
	$M=(M_{ij})$	$T=(T_{ijk})$





$$S(M)=\frac{1}{2}\mathrm{tr}M^2-\frac{\lambda}{3}\mathrm{tr}M^3$$

$$=\frac{1}{2}M_{ij}M_{ji}-\frac{\lambda}{3}M_{ij}M_{jk}M_{ki}$$

$$S(T)=\frac{1}{2}T_{ijk}T_{kji}$$

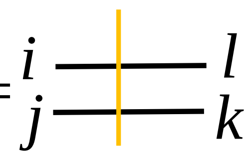
$$-\frac{\lambda}{4}T_{ijk}T_{klm}T_{mni}T_{jnl}$$

[Ambjørn-Durhuus-Jonsson, Sasakura 1991]

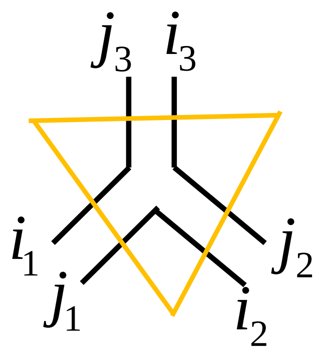
Strings and membranes (3′/3)

random surfaces	random volumes

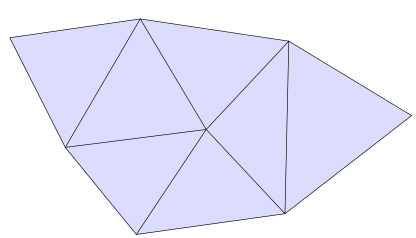
$$\left\langle M_{ij} M_{kl} \right\rangle_0 =$$



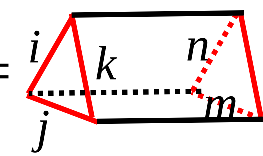
$$= \delta_{il}\delta_{jk}$$



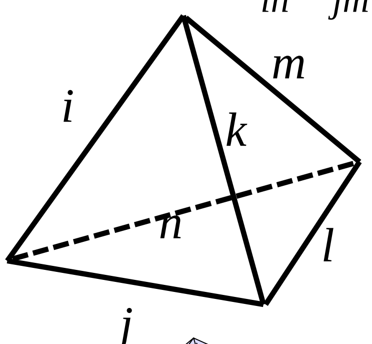
$$= \lambda\, \delta_{j_1 i_2} \delta_{j_2 i_3} \delta_{j_3 i_1}$$

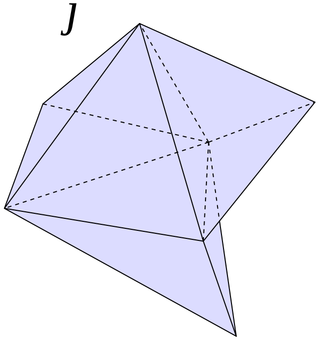


$$\left\langle T_{ijk} T_{lmn} \right\rangle_0 =$$



$$= \delta_{in}\delta_{jm}\delta_{kl}$$





Analytic solvability of matrix models

Diagonalization:

$$M = UxU^{-1} \quad \left(x = \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_N \end{pmatrix}, \quad U \in U(N) \right)$$

$$\Rightarrow \quad (dM) = \left(\prod_{i=1}^{N} dx_i \right) (dU) \prod_{i<j} \left(x_i - x_j \right)^2$$

Effective action:

$$Z = \int \left(\prod_i dx_i \right) e^{-S_{\text{eff}}(x)} \quad \textbf{with} \quad S_{\text{eff}}(x) \equiv S(x) - 2\sum_{i<j} \ln \left| x_i - x_j \right|$$

Large N analysis can be performed with the saddle point method

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But, no such analogues in the aforementioned tensor models.

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But, no such analogues in the aforementioned tensor models.

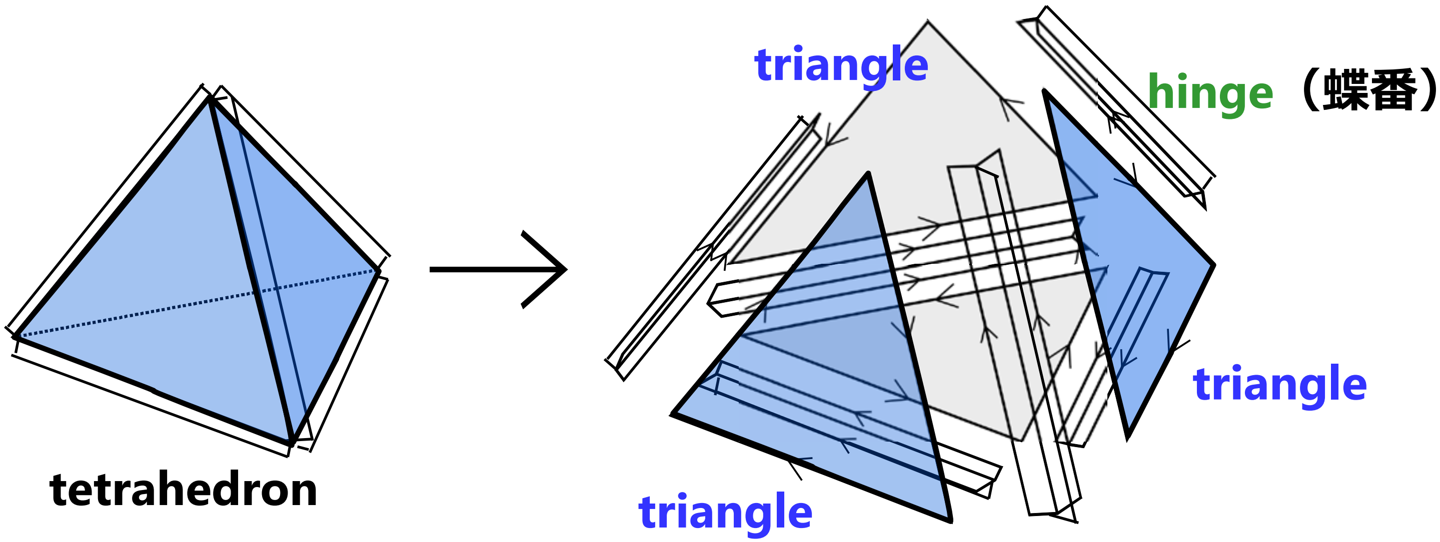
⇒ **We proposed a new class of "matrix models" which generate 3D random volumes.** **[MF-Sugishita-Umeda 2015]**

Main idea (1/2)

Adhere to using <u>triangles</u> (instead of tetrahedra) as building blocks.

We decompose a tetrahedral decomposition further to a collection of triangles which are glued together along hinges.

[cf: Chung-MF-Shapere 1994 for 3D TFT]



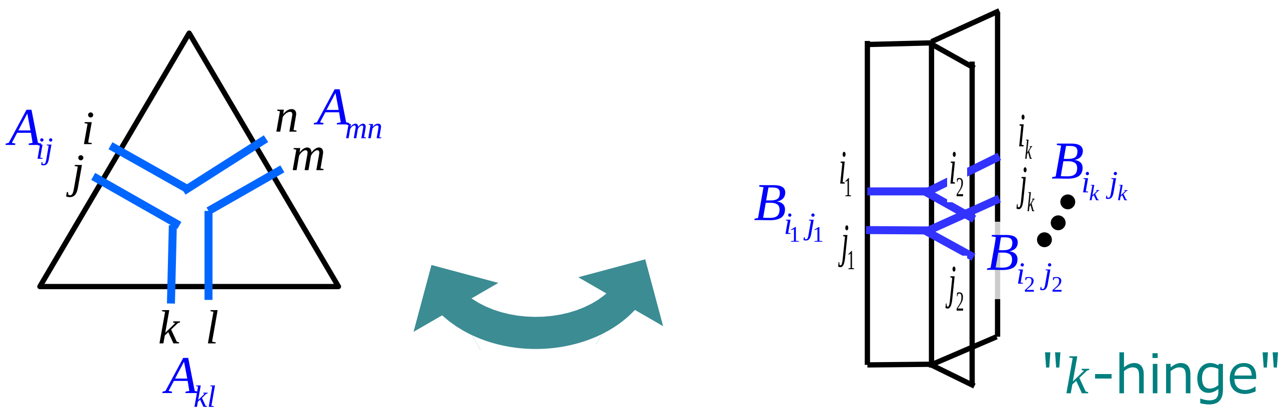
Main idea (2/2)

One can then take matrices as dynamical variables:

“triangle-hinge models”

[MF-Sugishita-Umeda 2015]

matrix A for triangles **matrix B for hinges**



connected with a propagator :

$$S(A,B) = A\cdot B - \underbrace{A\cdot A\cdot A}_{\text{triangle}} - \sum_k \underbrace{B\cdot B\cdot\cdots\cdot B}_{k\text{-hinge}}$$

Plan

0. Introduction (done)
1. The triangle-hinge models
   -- definition
   -- algebraic construction
2. General form of the free energy
   -- index network
   -- index functions
3. Matrix ring
   -- Feynman diagrams
   -- duality
4. Restricting to manifolds
5. Introducing matter fields
6. Recent developments
7. Conclusion and outlook

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Definition

Dynamical variables

$$A=\left(A_{ij}\right),\quad B=\left(B^{ij}\right)\quad \left(i,j=1,\dots,N\right)$$ **: $N\times N$ real symmetric matrices**

Action

$$S(A,B)=\frac{1}{2}A_{ij}B^{ij}-\frac{\lambda}{6}C^{(ij)(kl)(mn)}A_{ij}A_{kl}A_{mn}-\sum_{k\geq 3}\frac{\mu_k}{2k}Y_{(i_1 j_1)\cdots(i_k j_k)}B^{i_1 j_1}\cdots B^{i_k j_k}$$

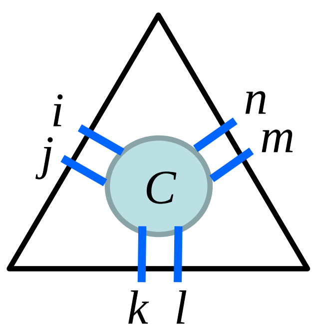
Feynman rules

[propagator]

$$\left\langle A_{ij}\,B^{kl}\right\rangle_0=\delta_i^k\delta_j^l+\delta_i^l\delta_j^k$$

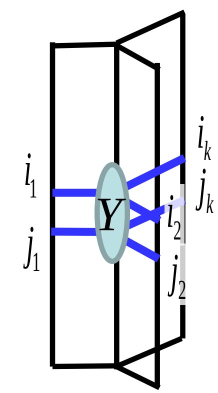
[vertices]

$$\lambda\, C^{(ij)(kl)(mn)}\sim$$



,

$$\mu_k\;Y_{(i_1 j_1)\cdots(i_k j_k)}\sim$$



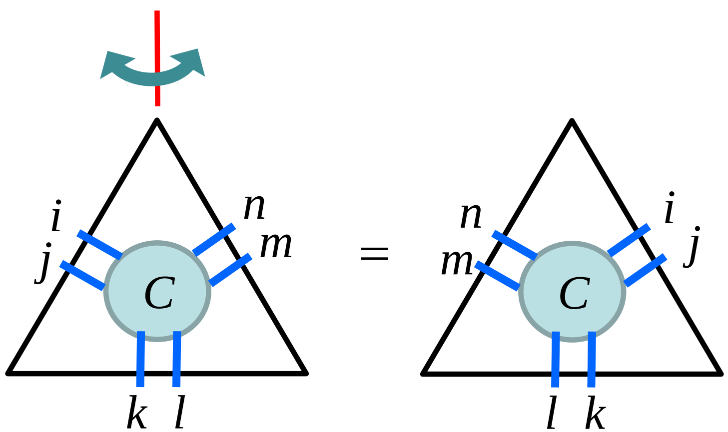
Symmetries of coupling constants

Cyclic symmetries:

$$C^{(ij)(kl)(mn)} = C^{(kl)(mn)(ij)}\,, \qquad Y_{(i_1 j_1)(i_2 j_2)\cdots(i_k j_k)} = Y_{(i_2 j_2)\cdots(i_k j_k)(i_1 j_1)}$$

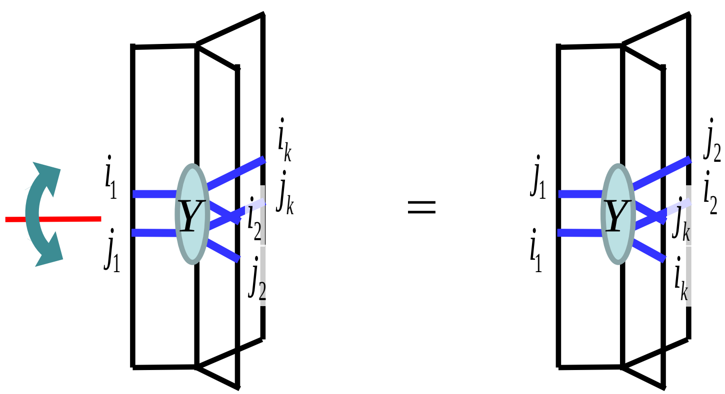
Flip symmetries:

(1)



$$C^{(ij)(kl)(mn)} = C^{(nm)(lk)(ji)}$$

(2)



$$Y_{(i_1 j_1)(i_2 j_2)\cdots(i_k j_k)} = Y_{(j_k i_k)\cdots(j_2 i_2)(j_1 i_1)}$$

Algebraic construction (1/3)

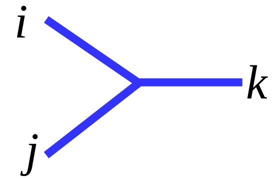
The couplings C and Y can be obtained from a semisimple associative algebra $\mathcal{A}$

NB: $\mathcal{A}$: associative algebra $\overset{\text{def}}{\Longleftrightarrow}$ $\mathcal{A}$: linear space with product "$\times$" satisfying associativity :

$$(a\times b)\times c = a\times(b\times c)$$

For a basis $\{e_i\}_{i=1,\ldots,N}$ $\left(\mathcal{A}=\bigoplus_i \mathbb{R}e_i\right)$, the product is expressed as

$$e_i\times e_j = y_{ij}{}^k e_k \quad \left(y_{ij}{}^k \text{ : structure constants}\right)$$



$= y_{ij}{}^k$

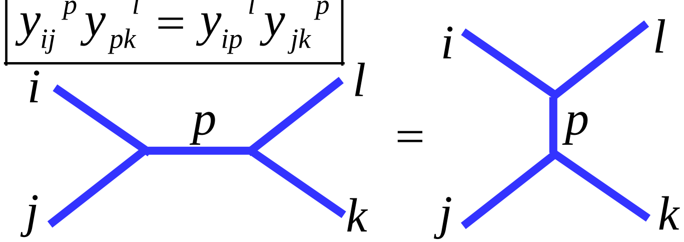
$\Downarrow$

associativity

$$(e_i\times e_j)\times e_k = e_i\times(e_j\times e_k)$$

$\Longleftrightarrow$ $\boxed{y_{ij}{}^p y_{pk}{}^l = y_{ip}{}^l y_{jk}{}^p}$

$\Longleftrightarrow$



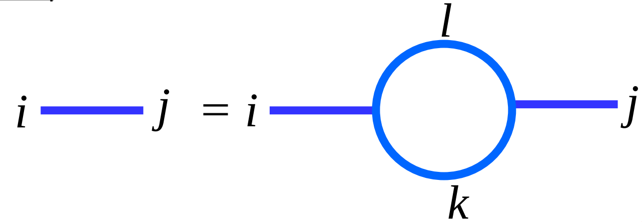
metric

$\boxed{g_{ij}\equiv y_{ik}{}^l y_{jl}{}^k}$

$\left[\begin{array}{l}\mathcal{A}\text{ : semisimple}\\ \Leftrightarrow\ g=(g_{ij})\text{ has } g^{-1}\equiv(g^{ij})\end{array}\right]$

[MF-Hosono-Kawai 1994]

$\Longleftrightarrow$

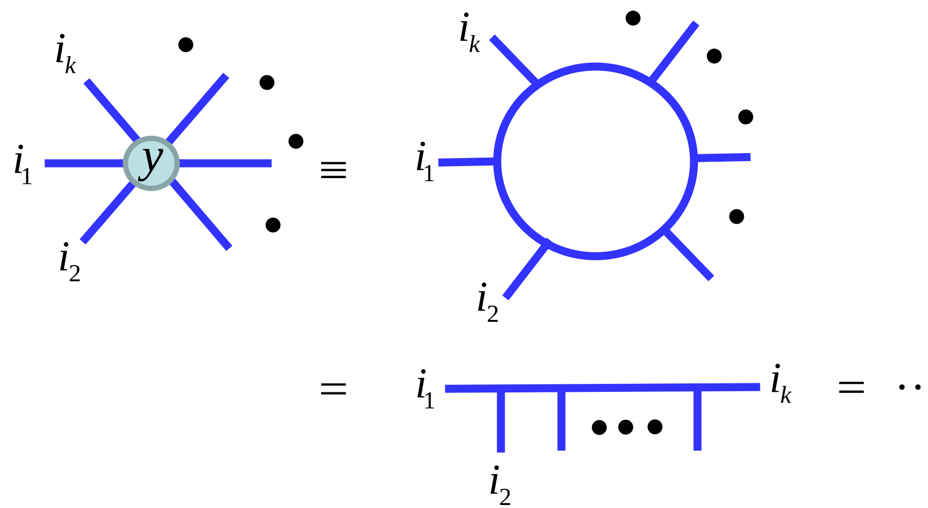


Algebraic construction (2/3)

We then introduce k-tensor:

$$\boxed{y_{i_1 i_2 \dots i_k} \equiv y_{i_1 j_1}{}^{j_k}\, y_{i_2 j_2}{}^{j_1} \cdots y_{i_k j_k}{}^{j_{k-1}}}$$

(cyclically symmetric)

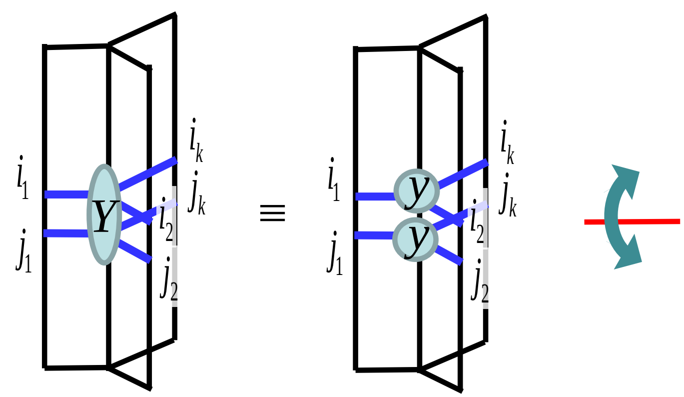


We define k-hinge tensor Y as:

$$\boxed{Y_{(i_1 j_1)\cdots(i_k j_k)} \equiv y_{i_1 i_2 \dots i_k}\, y_{j_k \dots j_2 j_1}}$$

(cyclically symmetric)

i.e.



This satisfies flip relation (2):

$$Y_{(i_1 j_1)(i_2 j_2)\cdots(i_k j_k)} = Y_{(j_k i_k)\cdots(j_2 i_2)(j_1 i_1)}$$

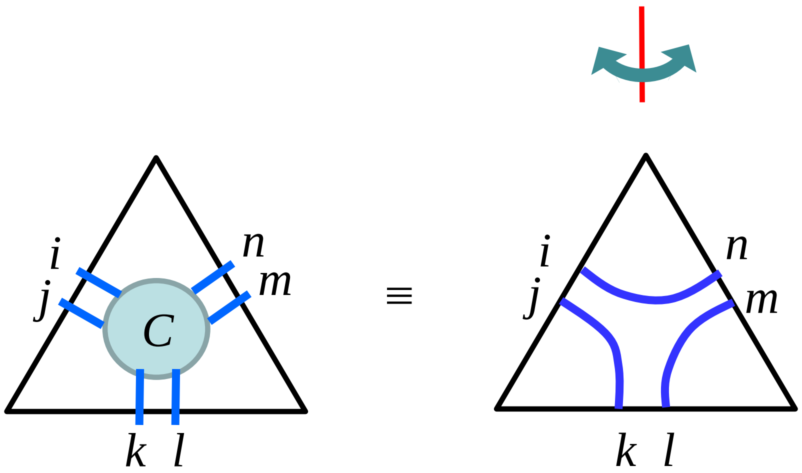
Algebraic construction (3/3)

We define triangle tensor C as:

$$\boxed{C^{(ij)(kl)(mn)} \equiv g^{jk} g^{lm} g^{ni}}$$

(cyclically symmetric)

i.e.



This satisfies flip relation (1):

$$C^{(ij)(kl)(mn)} = C^{(nm)(lk)(ji)}$$

but is not a unique solution to it.

(The latter fact is used
to restrict configurations to nonsingular ones
and to introduce matter fields into the TH models.)

Summary of the algebraic construction

Given a semisimple associative algebra $\mathcal{A} = \bigoplus_{1\le i\le N} \mathbb{R}\, e_i$

with structure constants $y_{ij}{}^{k} \quad \left(e_i \times e_j = y_{ij}{}^{k} e_k\right)$

Introduce $\begin{cases} k\textbf{-tensor:} & y_{i_1 i_2 \ldots i_k} \equiv y_{i_1 j_1}{}^{j_k}\, y_{i_2 j_2}{}^{j_1} \cdots y_{i_k j_k}{}^{j_{k-1}} \\ \textbf{metric:} & g_{ij} \equiv y_{ij} \;\Rightarrow\; (g^{ij}) \equiv (g_{ij})^{-1} \end{cases}$

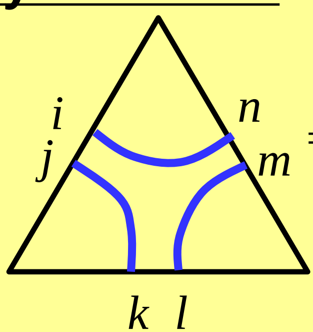
dynamical variables

$$A = \left(A_{ij}\right) = \left(A_{ji}\right), \quad B = \left(B^{ij}\right) = \left(B^{ji}\right) \quad (i, j = 1, \ldots, N)$$

action

$$S(A,B) = \frac{1}{2} A_{ij} B^{ij} - \frac{\lambda}{6} \overbrace{g^{jk} g^{lm} g^{ni}}^{C} A_{ij} A_{kl} A_{mn} - \sum_{k\ge 3} \frac{\mu_k}{2k} \overbrace{y_{i_1 \ldots i_k}\, y_{j_k \ldots j_1}}^{Y} B^{i_1 j_1} \cdots B^{i_k j_k}$$

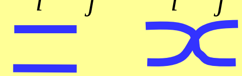
Feynman rules

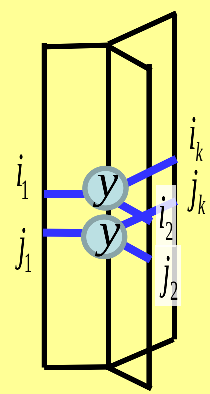


$= \lambda\, g^{jk} g^{lm} g^{ni}$

glued

$$\left\langle A_{ij} B^{kl} \right\rangle_0 = \delta_i^k \delta_j^l + \delta_i^l \delta_j^k$$





$= \mu_k\; y_{i_1 \ldots i_k}\, y_{j_k \ldots j_1}$

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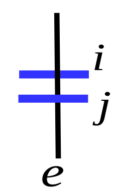
Perturbative expansion of free energy

$$S(A,B) = A\cdot B - \lambda C\,\underbrace{A\cdot A\cdot A}_{\text{triangle}} + \sum_k \mu_k Y\,\underbrace{B\cdot B\cdots\cdot B}_{k\text{-hinge}}$$

$$\log Z = \sum_\gamma \frac{1}{S(\gamma)} \sum_{\{(i,j)_e\}} \left\{ \prod_{f:\,\textbf{triangle}} \left[\lambda C^{(ij)(kl)(mn)}(f)\right] \prod_{h:\,\textbf{hinge}} \left[\mu_k Y_{(i_1 j_1)(i_2 j_2)\ldots(i_k j_k)}(h)\right] \right\}$$

$$= \sum_\gamma \frac{1}{S(\gamma)} \sum_{\{(i,j)_e\}} \left\{ \prod_{f:\,\textbf{triangle}} \left[\lambda g^{jk} g^{lm} g^{ni}\right] \prod_{h:\,\textbf{hinge}} \left[\mu_k\, y_{i_1 i_2 \ldots i_k}\, y_{j_k \ldots j_2 j_1}\right] \right\}$$

- γ : **connected diagram**
- $S(\gamma)$: **symmetry factor of** γ
- $(i,j)_e$: **indices on edge** e
- **Indices are contracted when two edges are identified**



Each triangle has a factor of λ, **and each** k**-hinge has a factor of** μ_k.

$$\boxed{\log Z = \sum_\gamma \frac{1}{S(\gamma)} \lambda^{s_2(\gamma)} \left(\prod_{k\geq 3} \mu_k^{s_1^k(\gamma)}\right) \mathcal{F}(\gamma)}$$

- $s_2(\gamma)$: **# of triangles (2-dim)**
- $s_1^k(\gamma)$: **# of** k**-hinges (1-dim)**

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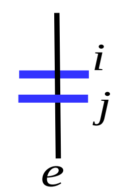

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- $s_2(\gamma)$: **# of triangles (2-dim)**
- $s_1^k(\gamma)$: **# of k-hinges (1-dim)**

$\mathcal{F}(\gamma)$: **function of** $y_{i_1\ldots i_k}$ **and** g^{ij} $\left(\textbf{thus a function of } y_{ij}{}^{k}\right)$

"index function of diagram γ"

Index functions and index network (1/3)

Properties of index function $\mathcal{F}(\gamma)$

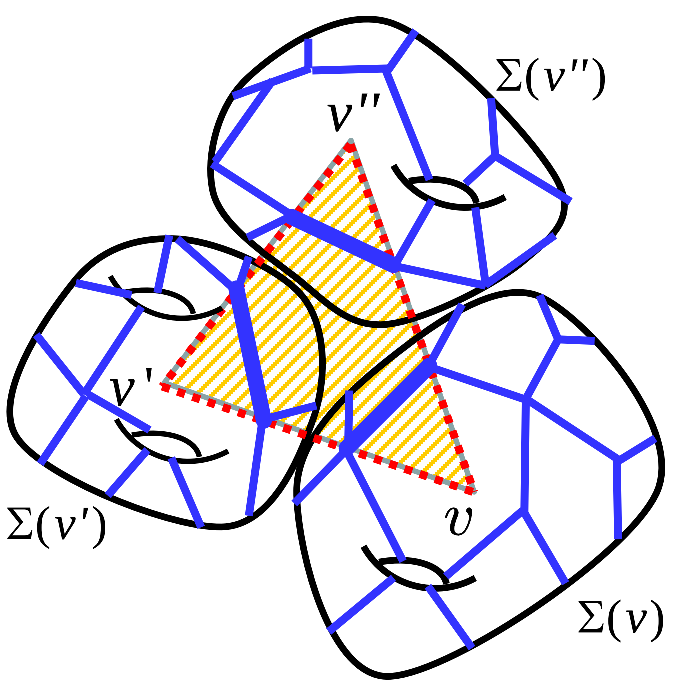
(1) Index networks form 2D closed surfaces $\Sigma(v)$ enclosing vertices v

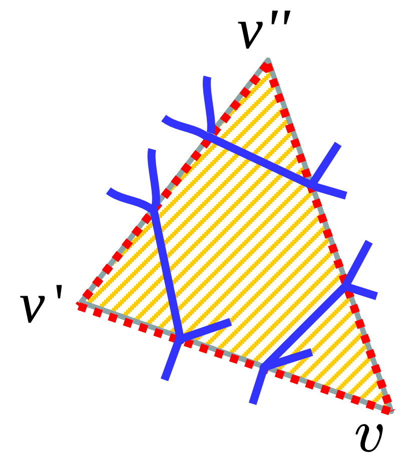
$\Rightarrow$ $\mathcal{F}(\gamma)$ is factorized as the product of the contributions from vertices v :

$$\mathcal{F}(\gamma) = \prod_{v\text{: vertex of }\gamma} \zeta(v)$$

(2) Each contribution $\zeta(v)$ is a 2D topological invariant of $\Sigma(v)$:

$$\zeta(v) = \mathcal{I}_{g(v)} \quad \left(g(v)\text{ : genus of }\Sigma(v)\right)$$





topological invariant $\mathcal{I}_{g(v)}$

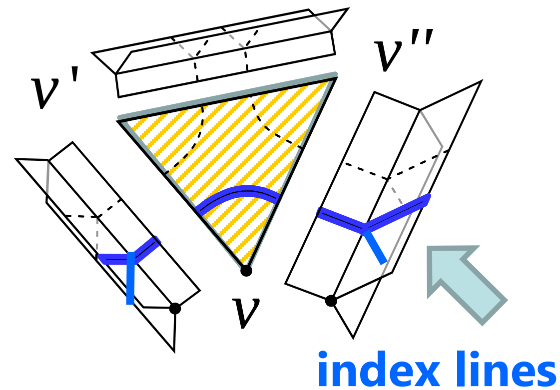
Index functions and index network (2/3)

(1) Index networks form 2D closed surfaces $\Sigma(v)$ enclosing vertices v

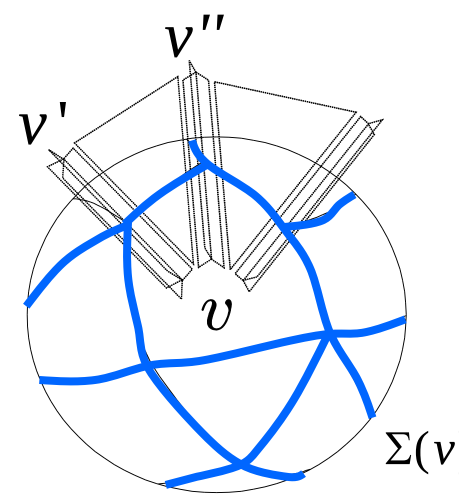
$(\because)$

Index lines on two different hinges
are connected (via intermediate triangles)
if and only if the hinges share the same vertex of γ.

(v)



The connected components of the index network
have a one-to-one correspondence to the vertices of γ,
and each connected component of
the index network can be regarded
as a <u>closed 2D surface</u> enclosing a vertex. ∎



$\Sigma(v)$ **: oriented, but not necessarily a sphere**

$\Rightarrow$ $\mathcal{F}(\gamma)$ **is factorized as** $\mathcal{F}(\gamma) = \prod_{v:\ \text{vertex of}\ \gamma} \zeta(v)$

Index functions and index network (3/3)

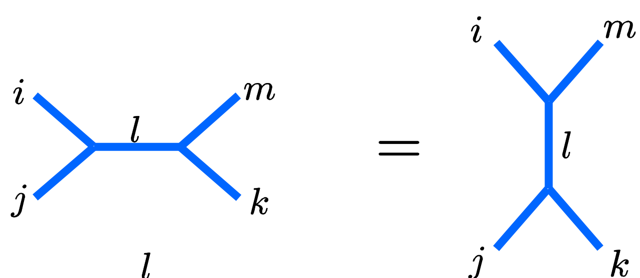
(2) Each contribution $\zeta(v)$ is a 2D topological invariant of $\Sigma(v)$:

$$\zeta(v) = \mathcal{I}_{g(v)}[\mathcal{A}] \qquad \big(g(v) \textbf{ : genus of } \Sigma(v)\big)$$

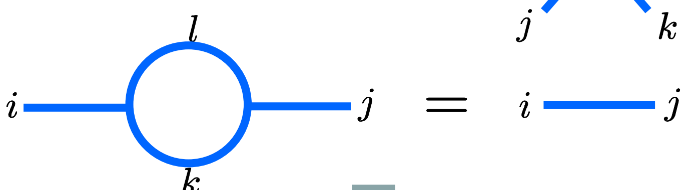
$(\because)$

$\zeta(v)$ for the index network around v is invariant under 2D topology-preserving local moves:

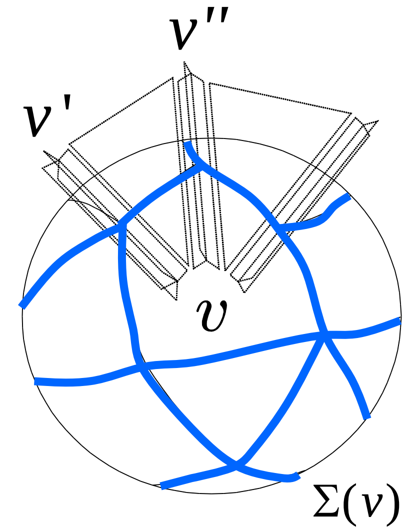
[MF-Hosono-Kawai 1994]



(associativity)



(definition of metric)

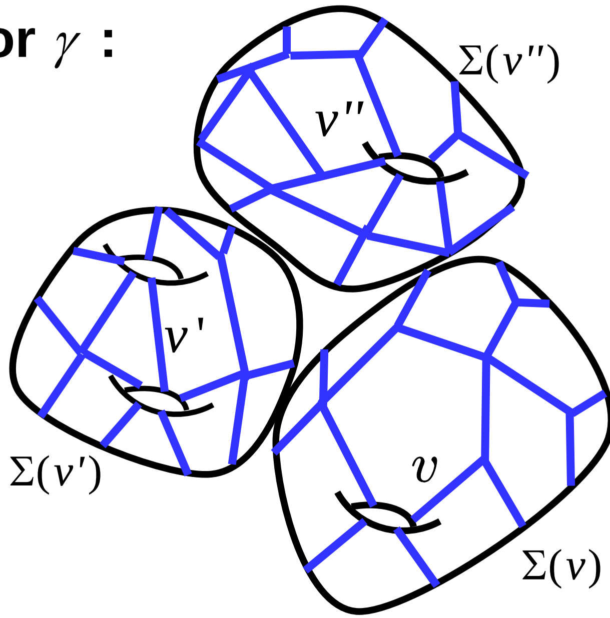


$\zeta(v)$ is the 2D topological invariant of $\Sigma(v)$ associated with $\mathcal{A}$:

$$\zeta(v) = \mathcal{I}_{g(v)}[\mathcal{A}] \quad \big(g(v) \textbf{ : genus of } \Sigma(v)\big)$$

Necessity of large N limit (1/2)

Index network for γ :



neighborhood of v
$=$ **3D cone** $\Sigma(v)\times\mathbb{R}_+$

(**NB:**
3D ball $B^3 = S^2\times\mathbb{R}_+$)

Important point:

In order for γ to give a nonsingular 3D configuration, each vertex must have a neighborhood of B^3 topology.

$\Rightarrow$ $g(v)=0$ **for** $\forall v\in\gamma$

$\Rightarrow$ **Large N limit !** **(to be shown in the next slide)**

Necessity of large N limit (2 / 2)

Recall:

$\mathcal{A}$: **semisimple** $\Leftrightarrow$ $\left\{ \begin{array}{l} \mathcal{A}\text{: a direct sum of matrix rings} \\ \boxed{\mathcal{A} = \oplus M_{n_k}(\mathbb{R})} \end{array} \right.$

(**NB:**
matrix ring $M_n(\mathbb{R}) \equiv \{ M = (M_{ab})$: $n \times n$ **matrix**$\}$
A more detailed explanation will be given shortly.)

We can easily find:

$$\boxed{\begin{aligned} N\left(= \dim \mathcal{A}\right) &= \sum_k n_k^2 \\ \mathcal{I}_g[\mathcal{A}] &= \sum_k n_k^{2-2g} \end{aligned}}$$

$\Downarrow$

$$\mathcal{F}(\gamma) = \prod_{v\text{: vertex of } \gamma} \left(\sum_k n_k^{2-2g(v)} \right)$$

$\Downarrow$

Every vertex has a neighborhood of B^3 topology.

$\Updownarrow$

We can single out the desired configurations $\left[g(v) = 0 \ \text{ for } \forall v \right]$
by taking the limit $n_k \to \infty$ for fixed number of vertices.

How to count # of vertices

We extend the algebra $\mathcal{A}$ to

$$\widehat{\mathcal{A}} \equiv \underbrace{\mathcal{A}\oplus\mathcal{A}\oplus\cdots\oplus\mathcal{A}}_{K\ \textbf{copies}} = K\mathcal{A}$$

Then, the index function has the form $\zeta\left(v;\widehat{\mathcal{A}}\right) = K\,\zeta\left(v;\mathcal{A}\right)$,

and thus the Boltzmann weight becomes

$$\frac{1}{S}\left[\prod_{v}\left[K\prod_{k\geq 3}(\lambda^{2}\mu_{k})^{\frac{1}{2}t_0^k(v)}\right]\left(\frac{n}{\lambda}\right)^{2-2g(v)}\left(\frac{1}{\lambda}\right)^{\frac{1}{3}d(v)}\right]$$

$$\left[\begin{array}{l} \left.\begin{array}{l} t_0^k(v):\ \textbf{\# of } k\textbf{-junctions} \\ t_1(v):\ \textbf{\# of segments} \\ t_2(v):\ \textbf{\# of polygons} \end{array}\right\}\ \textbf{in the index network around } v \\ d(v) \equiv 2t_1(v) - 3t_2(v) \end{array}\right]$$

By treating K as a free parameter,

one can count # of vertices.

Plan

0. Introduction
1. The triangle-hinge models
   -- definition
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   -- index functions
3. **Matrix ring**
   -- Feynman diagrams
   -- duality
4. Restricting to manifolds
5. Introducing matter fields
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Matrix ring (1/3)

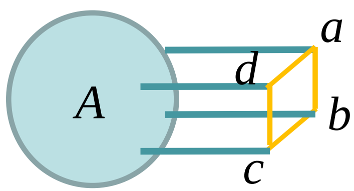
Set $\mathcal{A}$ to be a matrix ring:

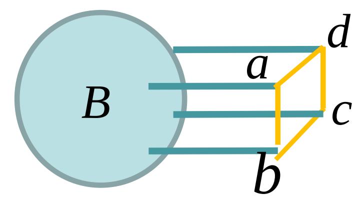
$$\mathcal{A} = M_n(\mathbb{R}) = \{M = (M_{ab}) \ \textbf{: } n\times n \ \textbf{matrix}\} = \bigoplus_{a,b=1}^{n} \mathbb{R}\, e_{ab}$$

$$e_{ab} = \begin{array}{cc} & b \\ a & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \ \textbf{: matrix unit}$$

$$\Longleftrightarrow \begin{pmatrix} i = (a,b) \\ N = n^2 \end{pmatrix}$$

variables :

$A_{ab,cd} =$ 

$B^{ab,cd} =$ 

thickened

From $e_{ab}\times e_{cd} = \delta_{bc} e_{ad} \equiv y_{ab,cd}{}^{ef} e_{ef}$ **, we have**

$$\begin{cases} y_{a_1b_1,a_2b_2,\ldots,a_kb_k} = n\,\delta_{b_1a_2}\delta_{b_2a_3}\cdots\delta_{b_ka_1}, \\ g^{ab,cd} = \dfrac{1}{n}\delta^{ad}\delta^{bc}, \\ C^{(a_1b_1,c_1d_1)(a_2b_2,c_2d_2)(a_3b_3,c_3d_3)} = \dfrac{1}{n^3}\delta^{d_1a_2}\delta^{d_2a_3}\delta^{d_3a_1}\delta^{b_3c_2}\delta^{b_2c_1}\delta^{b_1c_3} \end{cases}$$

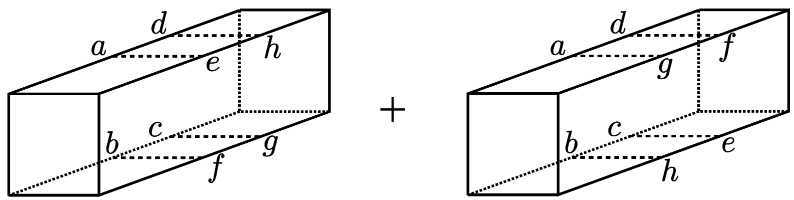
Matrix ring (2/3)

$$S(A,B)=\frac{1}{2}A_{ab,cd}B^{ab,cd}-\frac{\lambda}{6n^2}A_{ba,cd}A_{dc,ef}A_{fe,ab}-\sum_{k\geq 3}\frac{n^2\mu_k}{2k}B^{a_1a_2,b_2b_1}B^{a_2a_3,b_3b_2}\cdots B^{a_ka_1,b_1b_k}$$

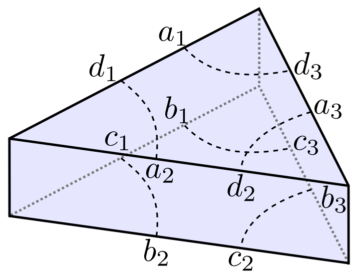
$$\left(A_{ab,cd}=A_{cd,ab},\quad B^{ab,cd}=B^{cd,ab}\right)$$

Feynman rules:

propagator:

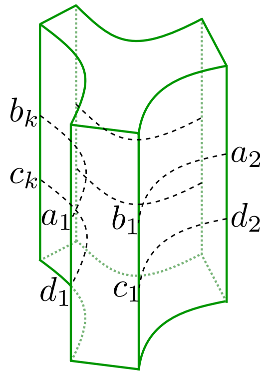


triangle:



$$=\frac{\lambda}{n^3}\delta^{d_1a_2}\delta^{d_2a_3}\delta^{d_3a_1}\delta^{b_3c_2}\delta^{b_2c_1}\delta^{b_1c_3}$$

k-hinge:

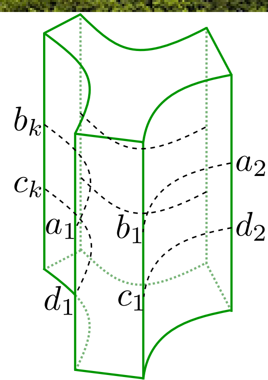


$$=n^2\mu_k\,\delta_{b_1a_2}\delta_{c_1d_2}\cdots\delta_{b_ka_1}\delta_{c_kd_1}$$

"New Otani vertex"

Matrix ring (2/3)

<u>k **-hinge:**</u>



$$= n^2 \mu_k \, \delta_{b_1 a_2} \delta_{c_1 d_2} \cdots \delta_{b_k a_1} \delta_{c_k d_1}$$

"New Otani vertex"

Matrix ring (2/3)

$$S(A,B)=\frac{1}{2}A_{ab,cd}B^{ab,cd}-\frac{\lambda}{6n^2}A_{ba,cd}A_{dc,ef}A_{fe,ab}-\sum_{k\geq 3}\frac{n^2\mu_k}{2k}B^{a_1a_2,b_2b_1}B^{a_2a_3,b_3b_2}\cdots B^{a_ka_1,b_1b_k}$$
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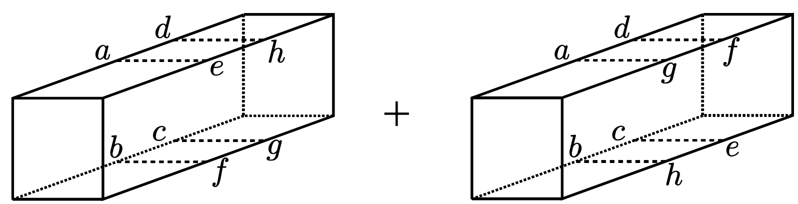
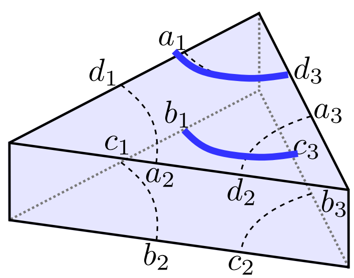
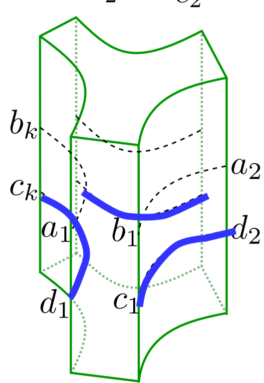


triangle:



$$=\frac{\lambda}{n^3}\delta^{d_1a_2}\delta^{d_2a_3}\delta^{d_3a_1}\delta^{b_3c_2}\delta^{b_2c_1}\delta^{b_1c_3}$$

k-hinge:



$$=n^2\mu_k\,\delta_{b_1a_2}\delta_{c_1d_2}\cdots\delta_{b_ka_1}\delta_{c_kd_1}$$

"New Otani vertex"

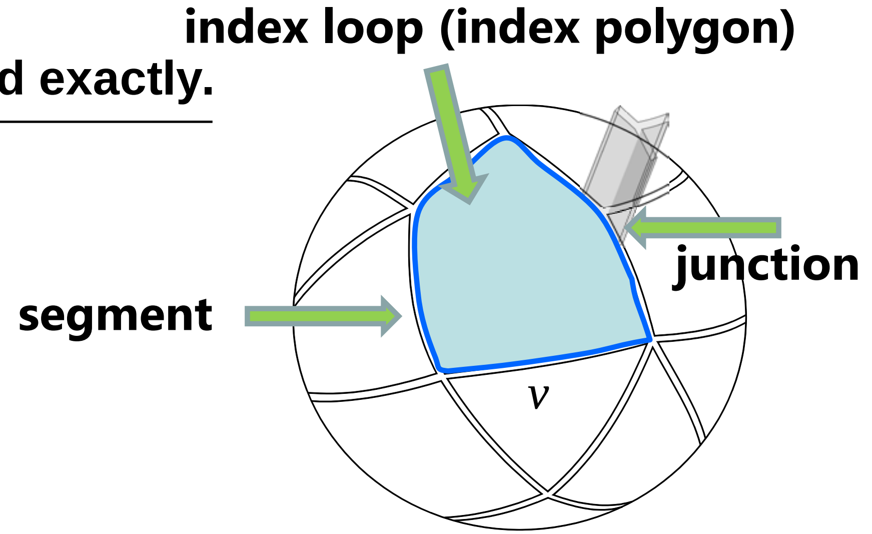
Matrix ring (3/3)

Index function $\zeta(v) = \mathcal{I}_{g(v)}$ can be calculated exactly.

- index loop $\Rightarrow \sum_{a=1}^{n} 1 = n$
- segment $\Rightarrow n^{-1}$ $\left(1/3 \text{ of triangle}\right)$
- junction $\Rightarrow n$ $\left(1/2 \text{ of hinge}\right)$

$$\zeta(v) = n^{\text{\#(polygon)-\#(segment)+\#(junction)}}$$

$$= \boxed{n^{2-2g(v)} \equiv \mathcal{I}_{g(v)}}$$



2D : $\Sigma(v)$

3D : cone $= \Sigma(v) \times \mathbb{R}_+$

Thus,

3D diagram $\Leftrightarrow$ a collection of 3D cones (each living at v)

In the limit $n \to \infty$ for fixed number of vertices,
the dominant contributions are from the 3D cone with genus 0:

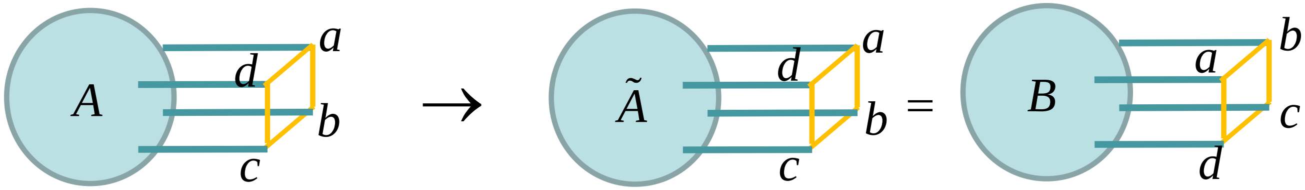
$S^2 \times \mathbb{R}_+ = B^3$ (3D ball)

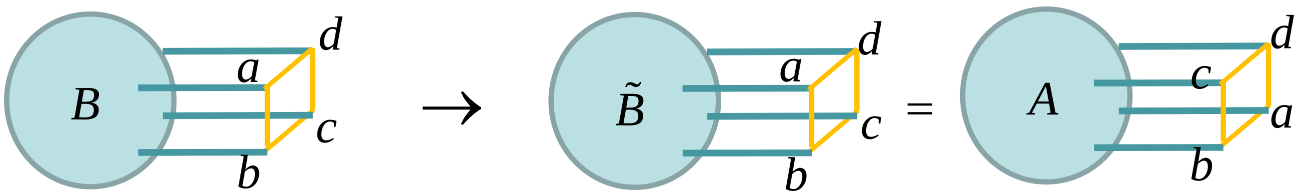
Duality (1/2)

Make a transformation of the form

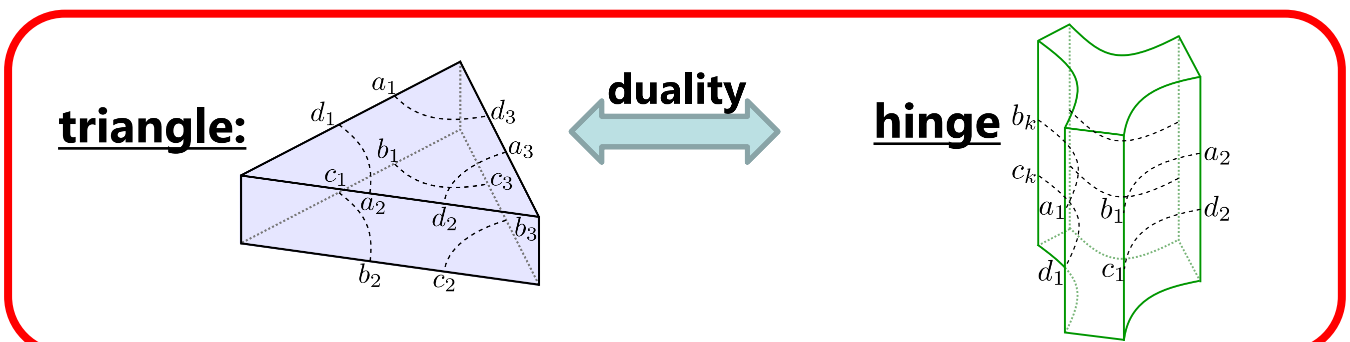
$$\begin{cases} A_{ab,cd} \to \tilde{A}_{ab,cd} \equiv n\, B^{bc,da} \\ B^{ab,cd} \to \tilde{B}^{ab,cd} \equiv n^{-1} A_{bc,da} \end{cases}$$

This rotates the quadruple of indices by 90 degrees:





This interchanges the role of triangles and hinges:



Duality (2/2)

The action transforms as

ℓ**-gon** $\qquad$ k**-hinge**

$$S(A,B)=\frac{1}{2}A_{ab,cd}B^{ab,cd}-\sum_{\ell\geq 3}\frac{\lambda_\ell}{2n^\ell \ell}A_{b_1a_1,a_2b_2}A_{b_2a_2,a_3b_3}\cdots A_{b_\ell a_\ell,a_1b_1}-\sum_{k\geq 3}\frac{n^2\mu_k}{2k}B^{a_1a_2,b_2b_1}B^{a_2a_3,b_3b_2}\cdots B^{a_ka_1,b_1b_k}$$

$$\left(A_{ab,cd}=A_{cd,ab},\quad B^{ab,cd}=B^{cd,ab}\right)$$

$\Downarrow$

k**-gon** $\qquad$ ℓ**-hinge**

$$\tilde{S}(\tilde{A},\tilde{B})=\frac{1}{2}\tilde{A}_{ab,cd}\tilde{B}^{ab,cd}-\sum_{k\geq 3}\frac{\mu_k}{2n^{k-2}k}\tilde{A}_{b_1a_1,a_2b_2}\tilde{A}_{b_2a_2,a_3b_3}\cdots\tilde{A}_{b_ka_k,a_1b_1}-\sum_{\ell\geq 3}\frac{\lambda_\ell}{2\ell}\tilde{B}^{a_1a_2,b_2b_1}\tilde{B}^{a_2a_3,b_3b_2}\cdots\tilde{B}^{a_\ell a_1,b_1b_\ell}$$

$$\left(\tilde{A}_{ab,cd}=\tilde{A}_{cd,ab},\quad \tilde{B}^{ab,cd}=\tilde{B}^{cd,ab}\right)$$

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ω method (1/3)

We consider a matrix ring with $n=3m$ **:**

$$\mathcal{A}_{\textbf{grav}} \equiv M_{n=3m}(\mathbb{R}) = \{M=(M_{ab}) : n\times n \textbf{ matrix}\} \Longleftrightarrow \begin{pmatrix} i=(a,b) \\ N=n^2=(3m)^2 \end{pmatrix}$$

$$= \bigoplus_{a,b} \mathbb{R}\, e_{ab}$$

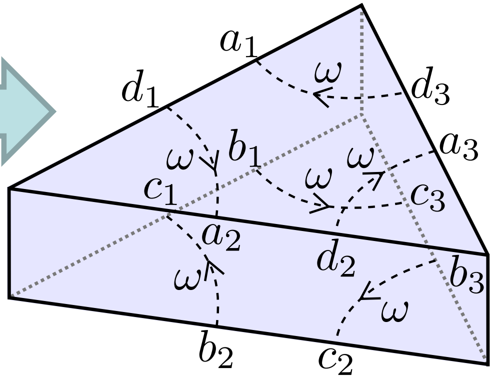
and change the form of C

from $C^{(a_1b_1,c_1d_1)(a_2b_2,c_2d_2)(a_3b_3,c_3d_3)} = \frac{1}{n^3}\delta^{d_1a_2}\delta^{d_2a_3}\delta^{d_3a_1}\delta^{b_3c_2}\delta^{b_2c_1}\delta^{b_1c_3}$

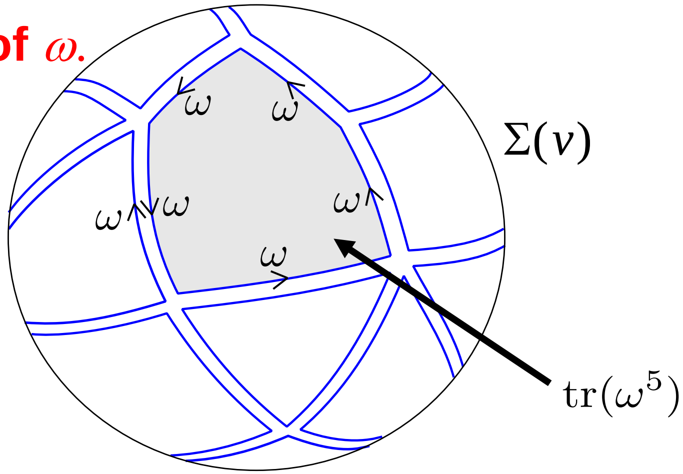
to : $C_{\textbf{grav}}^{(a_1b_1,c_1d_1)(a_2b_2,c_2d_2)(a_3b_3,c_3d_3)} \equiv \frac{1}{n^3}\omega^{d_1a_2}\omega^{d_2a_3}\omega^{d_3a_1}\omega^{b_3c_2}\omega^{b_2c_1}\omega^{b_1c_3}$

$$\left[\ \omega \equiv \begin{pmatrix} 0 & 1_m & 0 \\ 0 & 0 & 1_m \\ 1_m & 0 & 0 \end{pmatrix}\ \right]$$

This changes the triangle to :



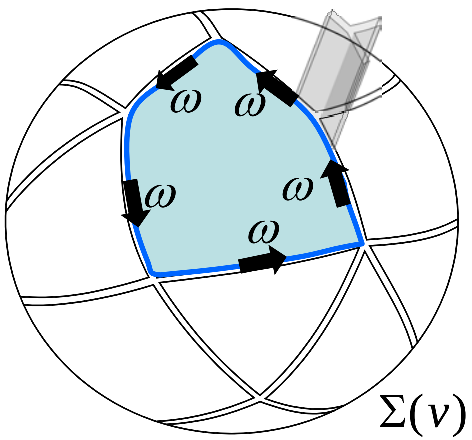
Every segment gets an insertion of ω.



ω method (2/3)

Each polygon with ℓ segments gets a factor:

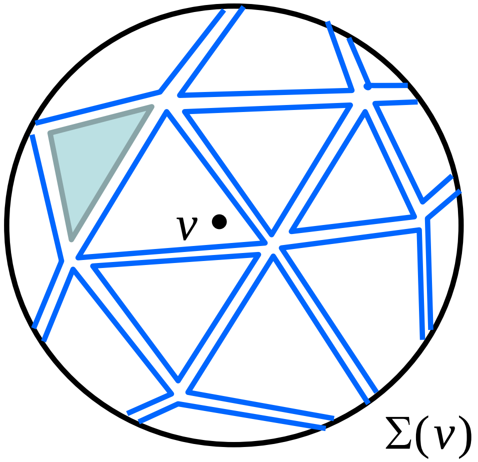
$$\text{tr}\ \omega^{\ell} = \begin{cases} n = 3m & (\ell = 0 \ \ \textbf{mod}\ 3) \\ 0 & (\ell \neq 0 \ \ \textbf{mod}\ 3) \end{cases}$$



Furthermore, if we take the limit

$$\boxed{n \to \infty \ \ \textbf{with}\ \ \frac{n}{\lambda}\ \ \textbf{and}\ n^2 \mu_k\ \ \textbf{kept finite}},$$

the dominant contribution comes from diagrams with $\ell = 3$. $\cdots\cdots$ **($*$)**



ω method (2/3)

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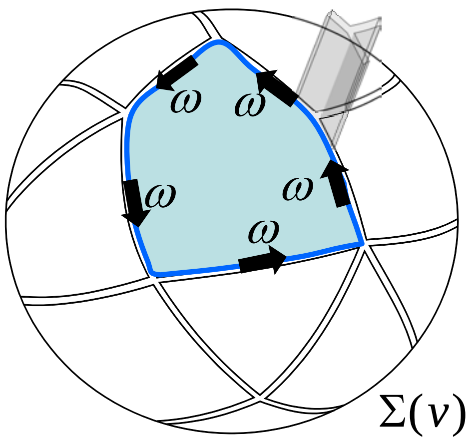

Furthermore, if we take the limit

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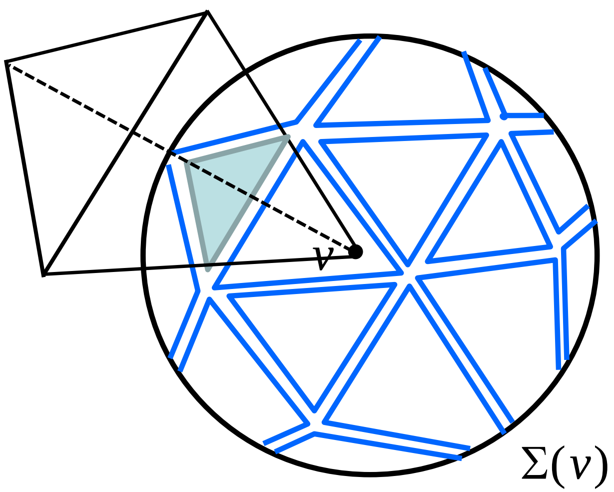
the dominant contribution comes from diagrams with $\ell = 3$. $\cdots\cdots$ **(∗)**

⇩

Every vertex v is a vertex of a tetrahedron.

⇩



The dominant diagrams are such that

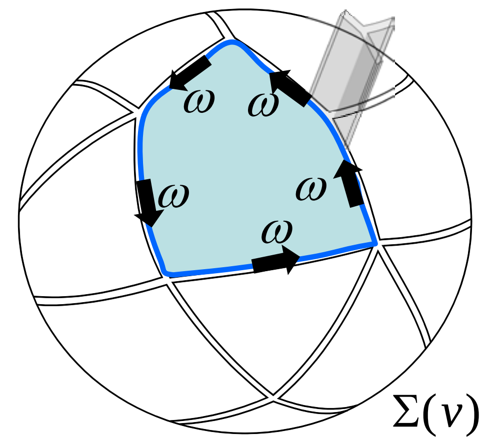
(1) the triangles and hinges form tetrahedra

(2) neighborhoods around each vertex have the topology of B^3

ω method (2/3)

Each polygon with ℓ segments gets a factor:

$$\text{tr}\ \omega^{\ell} = \begin{cases} n = 3m & (\ell = 0 \ \ \textbf{mod}\ 3) \\ 0 & (\ell = 0 \ \ \textbf{mod}\ 3) \end{cases}$$



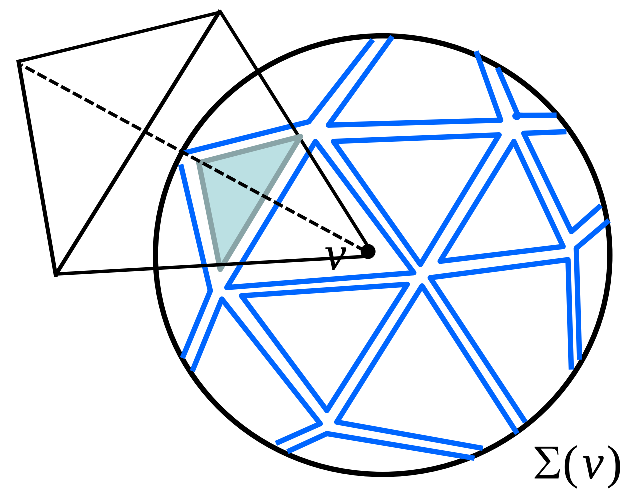
Furthermore, if we take the limit

$\boxed{n \to \infty \ \ \textbf{with} \ \ \dfrac{n}{\lambda} \ \ \textbf{and} \ n^2\mu_k \ \textbf{kept finite}}$,

the dominant contribution comes from diagrams with $\ell = 3$. $\cdots\cdots$ **(∗)**

⇩

Every vertex v is a vertex of a tetrahedron.



⇩

The dominant diagrams are such that

(1) the triangles and hinges form tetrahedra

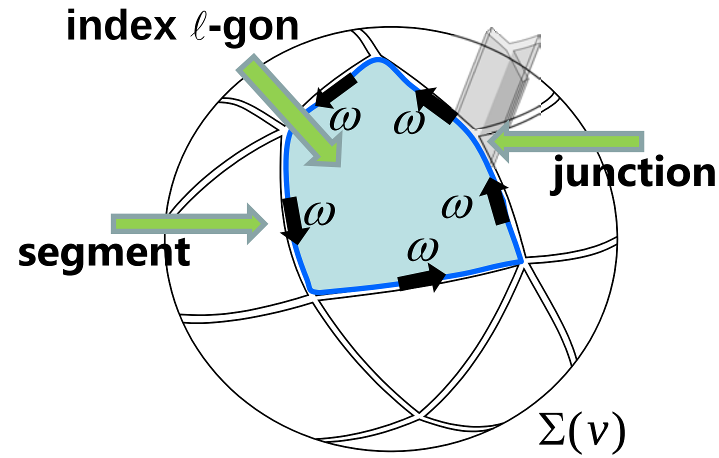
(2) neighborhoods around each vertex

have the topology of B^3

tetrahedral decomposition of 3d manifold!

ω method (3/3)

Proof of $(*)$



We first recall $\ell = 3\bar{\ell}$ $(\bar{\ell} \geq 1)$ and set

$$\begin{cases} t_2^\ell(v) : \text{\# of index } \ell\text{-gons} \\ t_1(v) : \text{\# of segments} \\ t_0^k(v) : \text{\# of } k\text{-junctions} \end{cases} \qquad \begin{cases} s_2 : \text{\# of triangles in } \gamma \\ s_1^k : \text{\# of } k\text{-hinges in } \gamma \\ s_0 : \text{\# of vertices in } \gamma \end{cases}$$

Due to the relations

$$s_2 = \frac{1}{3}\sum_v t_1(v), \quad s_1^k = \frac{1}{2}\sum_v t_0^k(v), \quad 2t_1(v) = \sum_{\bar{\ell}\geq 1} \ell\, t_2^\ell(v),$$

the Boltzmann weight is expressed as

$$\frac{1}{S}\lambda^{s_2}\left(\prod_{k\geq 3}\mu_k^{s_1^k}\right)\prod_v n^{2-2g(v)} = \frac{1}{S}\left[\prod_v\left[\prod_{k\geq 3}(\lambda^2\mu_k)^{\frac{1}{2}t_0^k(v)}\right]\left(\frac{n}{\lambda}\right)^{2-2g(v)}\left(\frac{1}{\lambda}\right)^{\frac{1}{3}d(v)}\right]$$

Here, the function $d(v)$ is given by

$$d(v) = 2t_1(v) - 3\sum_{\ell\geq 3} t_2^\ell(v) = \sum_{\ell \geq 3}(\ell - 3)t_2^\ell(v) \geq 0$$

We thus see that if we take the limit $\lambda \to \infty$ with $\lambda^2\mu_k$ and n/λ kept finite

(or equivalently, the limit $n \to \infty$ with $n^2\mu_k$ and n/λ kept finite),

the leading contribution comes from the diagrams with $d(v) = 0$ (i.e. with $\ell = 3$). ∎

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Generalities

So far, we have only discussed "gravity sector":

$$\mathcal{A}_{\textbf{grav}} = M_{n=3m}(\mathbb{R}) = \bigoplus_{1\le a,b\le n} \mathbb{R}\, e_{ab} \quad \left(n = 3m \text{ : the size of matrices}\right)$$

$$C_{\textbf{grav}} = \left(C_{\textbf{grav}}^{(a_1b_1,c_1d_1)(a_2b_2,c_2d_2)(a_3b_3,c_3d_3)} = \frac{1}{n^3}\,\omega^{d_1a_2}\,\omega^{d_2a_3}\,\omega^{d_3a_1}\,\omega^{b_3c_2}\,\omega^{b_2c_1}\,\omega^{b_1c_3} \right) \quad \left[\ \omega \equiv \begin{pmatrix} 0 & 1_m & 0 \\ 0 & 0 & 1_m \\ 1_m & 0 & 0 \end{pmatrix} \right]$$

Restriction to manifolds still works when we extend the system as

$$\mathcal{A} \equiv \mathcal{A}_{\textbf{grav}} \otimes \mathcal{A}_{\textbf{matt}}$$

$$C \equiv C_{\textbf{grav}} \otimes C_{\textbf{matt}}$$

$\Downarrow$

The index function of diagram γ factorizes as

$$\mathcal{F}(\gamma) \equiv \mathcal{F}(\gamma;\mathcal{A}) = \mathcal{F}(\gamma;\mathcal{A}_{\textbf{grav}})\,\mathcal{F}(\gamma;\mathcal{A}_{\textbf{matt}})$$

$$\equiv \mathcal{F}_{\textbf{grav}}(\gamma)\,\mathcal{F}_{\textbf{matt}}(\gamma)$$

We wil find that

one can assign local matter d.g.f. to simplices of arbitrary dimensions

(tetrahedra, triangles, edges, vertices)

by taking $(\mathcal{A}_{\textbf{matt}}, C_{\textbf{matt}})$ appropriately.

Coloring tetrahedra (1/2)

[$A_{\alpha\beta,\gamma\delta}$ is actually $A_{\underbrace{ab,cd}_{\text{grav}};\underbrace{\alpha\beta,\gamma\delta}_{\text{matt}}}$ (we omit gravity part)]

We take

$$\mathcal{A}_{\text{matt}} = M_q(\mathbb{R}) = \bigoplus_{1\le\alpha,\beta\le q} \mathbb{R}\, e_{\alpha\beta}$$

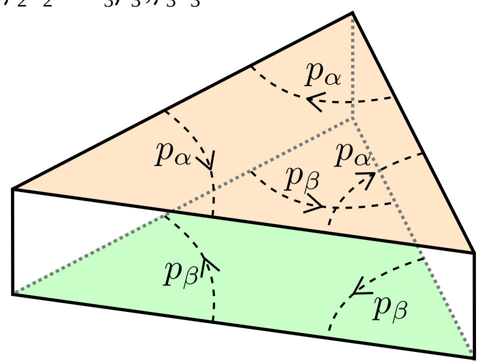
$$C_{\text{matt}} = \left\{ C_{\text{matt}}^{(\alpha_1\beta_1,\gamma_1\delta_1)(\alpha_2\beta_2,\gamma_2\delta_2)(\alpha_3\beta_3,\gamma_3\delta_3)} = \frac{1}{n^3}\, p_\alpha^{\delta_1\alpha_2} p_\alpha^{\delta_2\alpha_3} p_\alpha^{\delta_3\alpha_1} p_\beta^{\beta_3\gamma_2} p_\beta^{\beta_2\gamma_1} p_\beta^{\beta_1\gamma_3} \right\}$$

$$\left(p_\alpha = \begin{array}{cc} & \alpha \\ \alpha & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array} \quad \left(p_\alpha^{\alpha_1\alpha_2} \equiv \delta_\alpha^{\alpha_1}\delta_\alpha^{\alpha_2} \right) \right)$$

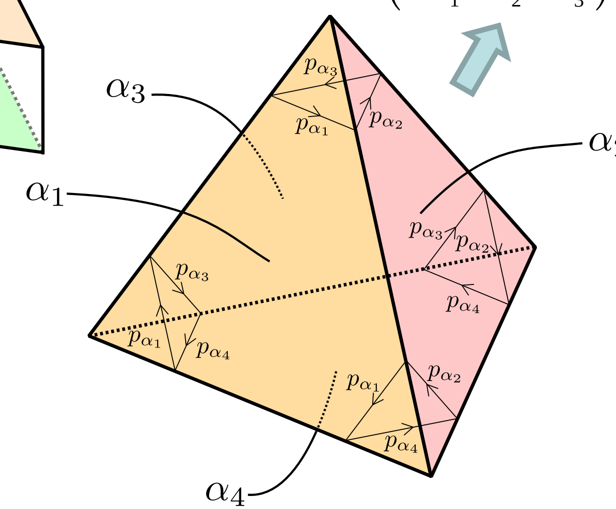
which gives the interaction for triangles:

$$-\sum_{\alpha,\beta=1}^{q} \frac{\lambda_{\alpha\beta}}{6q^2} \sum_{\alpha_1,\dots,\delta_3=1}^{q} A_{\alpha_1\beta_1,\gamma_1\delta_1} A_{\alpha_2\beta_2,\gamma_2\delta_2} A_{\alpha_3\beta_3,\gamma_3\delta_3}\, p_\alpha^{\delta_1\alpha_2} p_\alpha^{\delta_2\alpha_3} p_\alpha^{\delta_3\alpha_1} p_\beta^{\beta_3\gamma_2} p_\beta^{\beta_2\gamma_1} p_\beta^{\beta_1\gamma_3}$$

This changes the triangles to



$\mathbf{tr}\left(p_{\alpha_1} p_{\alpha_2} p_{\alpha_3} \right)$



The four index triangles in each tetrahedron gives the factor

$$\mathbf{tr}\left(p_{\alpha_1}p_{\alpha_2}p_{\alpha_3}\right)\mathbf{tr}\left(p_{\alpha_2}p_{\alpha_1}p_{\alpha_4}\right)\mathbf{tr}\left(p_{\alpha_1}p_{\alpha_3}p_{\alpha_4}\right)\mathbf{tr}\left(p_{\alpha_3}p_{\alpha_2}p_{\alpha_4}\right)$$

$$= \begin{cases} 1 & (\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4) \\ 0 & (\text{otherwise}) \end{cases}$$

Each tetrahedron has a single color.

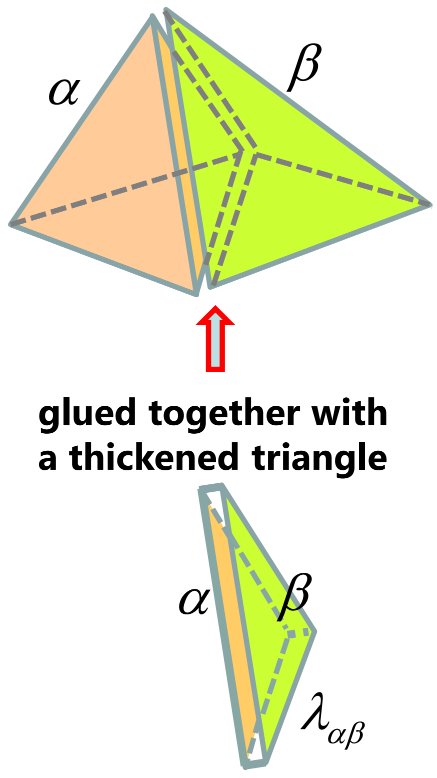
Coloring tetrahedra (2/2)

Feynman rules

(1) For a fixed tetrahedral decomposition,
we assign a color $\alpha = 1, \ldots, q$ **to each tetrahedron.**

(2) If two adjacent tetrahedra have colors α **and** β**,**
we multiply $\lambda_{\alpha\beta}$.

(3) Sum over tetrahedral decompotitions.



q**-state spin system on 3D random volumes**

Example:

$q = 2 \Longleftrightarrow$ **3D QG coupled to the Ising model**

Comments

(1) One can assign local matter d.o.f. to simplices of arbitrary dimensions
(tetrahedra, triangles, edges, vertices).

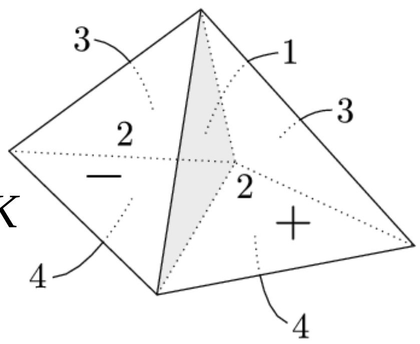
(2) (A version of) 3D colored tensor models
can be realized as a triangle-hinge model
by making coloring for tetrahedra, triangles and edges at a time.

Relation to colored tensor models (1/2)

(A version of) 3D Colored Tensor Models are given by the action

$$S=\sum_{\mu=1}^{4}\phi^{\mu}_{IJK}\bar{\phi}^{\mu}_{IJK}+\kappa\phi^{1}_{IJK}\phi^{2}_{KML}\phi^{3}_{MJN}\phi^{4}_{LNI}+\kappa\bar{\phi}^{1}_{IJK}\bar{\phi}^{2}_{KML}\bar{\phi}^{3}_{MJN}\bar{\phi}^{4}_{LNI}$$

Here, $\begin{cases}\{I\}\text{: color index set (assigned to edges)}\\ \{\mu\}=\{1,2,3,4\}\text{ (assigned to triangles)}\\ \phi^{\mu}_{IJK}\text{ and }\bar{\phi}^{\mu}_{IJK}\text{ have no symmetry properties w.r.t. }I,J,K\end{cases}$



⇩

Feynman rules

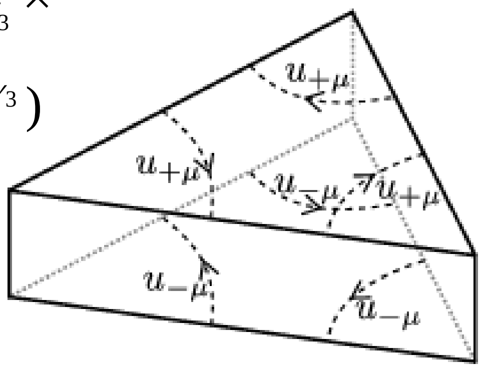
- Interaction vertices are represented by two types of tetrahedra $(\alpha=\pm)$ and any two adjacent tetrahedra have different types
- 4 different colors $\mu=1,\dots,4$ are assigned to 4 triangles of each tetrahedron, s.t. the assignment agrees with the orientation of the tetrahedra when $\alpha=+$ while it is opposite when $\alpha=-$
- 2 tetrahedra are glued at their faces in such a way that
  (1) 2 triangles to be identified have the same color μ and
  (2) 2 edges to be identified have the same pair of colors $(\mu\nu)$

Relation to colored tensor models (2/2)

The above model can be realized as a triangle-hinge model by setting the matter associative algebra to $\mathcal{A}_{\text{mat}} = M_{2s}(\mathbb{R})$ **and by coloring tetrahedra, triangles and edges <u>at a time</u>**

<u>action</u> (only writing down the matter sector):

$$S = \sum_{(\mu\nu)} \frac{1}{2} A^{(\mu\nu)}_{\alpha\beta\gamma\delta} B^{\alpha\beta\gamma\delta}_{(\mu\nu)} - \frac{\lambda}{6(2s)^3} \sum_{\mu=1}^{4} \frac{1}{6} \sum_{\nu,\rho,\sigma=1}^{4} A^{(\mu\nu)}_{\alpha_1\beta_1\gamma_1\delta_1} A^{(\mu\nu)}_{\alpha_2\beta_2\gamma_2\delta_2} A^{(\mu\nu)}_{\alpha_3\beta_3\gamma_3\delta_3} \times$$
$$\times \left(u_{+\mu}^{\delta_1\alpha_2} u_{+\mu}^{\delta_2\alpha_3} u_{+\mu}^{\delta_3\alpha_1} u_{-\mu}^{\beta_3\gamma_2} u_{-\mu}^{\beta_2\gamma_1} u_{-\mu}^{\beta_1\gamma_3} + u_{-\mu}^{\delta_1\alpha_2} u_{-\mu}^{\delta_2\alpha_3} u_{-\mu}^{\delta_3\alpha_1} u_{+\mu}^{\beta_3\gamma_2} u_{+\mu}^{\beta_2\gamma_1} u_{+\mu}^{\beta_1\gamma_3} \right)$$
$$- \sum_{k\geq 2} \frac{n^2 \mu_k}{2k} \sum_{(\mu\nu)} B^{\alpha_1\alpha_2\beta_2\beta_1}_{(\mu\nu)} B^{\alpha_2\alpha_3\beta_3\beta_2}_{(\mu\nu)} \cdots B^{\alpha_k\alpha_1\beta_1\beta_k}_{(\mu\nu)}$$



Here,

- $(\mu,\nu) = (\nu,\mu)$ $(\mu,\nu = 1,\ldots,4;\ \mu \neq \nu)$ **stand for the colors assinged to edges**
- $u_{+\mu} = \begin{pmatrix} u_\mu & 0 \\ 0 & 0 \end{pmatrix}$, $u_{-\mu} = \begin{pmatrix} 0 & 0 \\ 0 & u_\mu{}^T \end{pmatrix}$

<u>e.g.</u> $(s=6)$

$$u_1 = 2^{-2/3}\begin{pmatrix} 0 & \sigma_1 & 0 \\ 0 & 0 & \sigma_1 \\ 0 & 0 & 0 \end{pmatrix},\ u_2 = 2^{-2/3}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sigma_2 \\ -\sigma_2 & 0 & 0 \end{pmatrix},\ u_3 = 2^{-2/3}\begin{pmatrix} 0 & \sigma_3 & 0 \\ 0 & 0 & 0 \\ \sigma_3 & 0 & 0 \end{pmatrix},\ u_4 = 2^{1/3}\begin{pmatrix} 0 & i\sigma_2 & 0 \\ 0 & 0 & -\sigma_3 \\ -\sigma_1 & 0 & 0 \end{pmatrix}$$

where $s \times s$ **matrix** u_μ **(**s **can be arbitrary) are chosen so as to satisfy**

$$\mathrm{tr}(u_\mu u_\nu u_\rho)\ \mathrm{tr}(u_\nu u_\mu u_\sigma)\ \mathrm{tr}(u_\mu u_\rho u_\sigma)\ \mathrm{tr}(u_\rho u_\nu u_\sigma) = \begin{cases} 1 & (\epsilon_{\mu\nu\rho\sigma} = +1) \\ 0 & (\text{otherwise}) \end{cases}$$

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Possible 3$^{\text{rd}}$ order phase transition

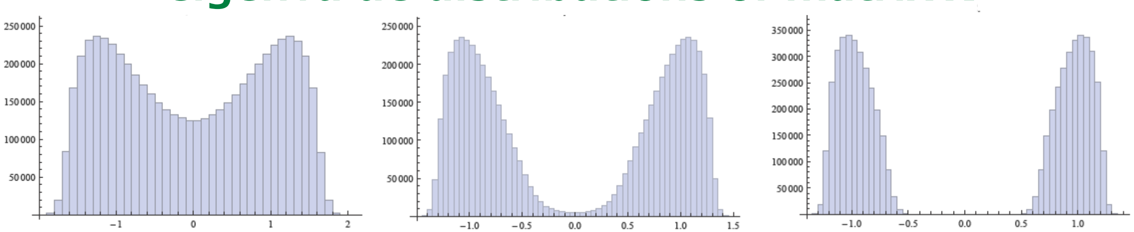
As a recent development, we have shown:

- **There exists an integration contour for which the free energy is defined nonperturbatively.**
- **The models can almost be reduced to a dynamical system of eigenvalues.**

$$S(A,B)=\beta\Big[\underbrace{K(A,B)}_{\text{bilinear}}+\sum_{\ell}\mathrm{tr}A^{\ell}+\sum_{k}\mathrm{tr}B^{k}\Big]$$

- **The models exhibit 3$^{\text{rd}}$-order phase transitions in the absence of ω, that are**
  - **-- "1-cut to 2-cut" phase transitions**
  - **-- very similar to 1-matix models with a double-well potential**

eigenvalue distributions of matrix A

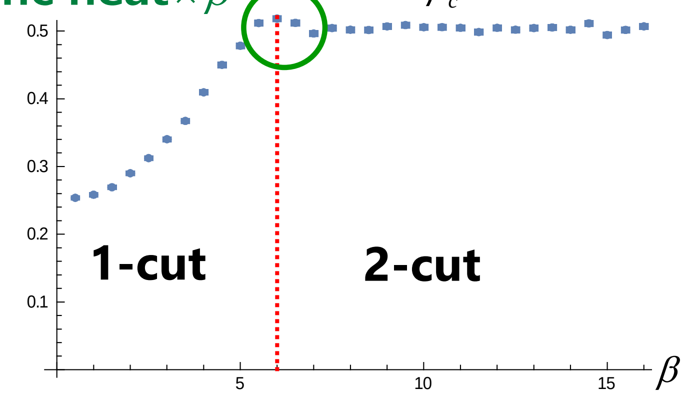


$\beta=1$ $\beta=6$ $\beta=12$

(1-cut) ⟷ **(2-cut)**

specific heat $\times\beta^2$

3$^{\text{rd}}$-order phase transition at $\beta_c\sim 6$



Road to the manifold case (1/2)

When one introduces the nontrivial ω, $\omega = \begin{pmatrix} 0 & 1_m & 0 \\ 0 & 0 & 1_m \\ 1_m & 0 & 0 \end{pmatrix}$, **to restrict to manifolds,**

the matrix bilinear $K(A,B)$ **becomes complex after the change of variables:**

$$S(A,B) = \frac{1}{2} K_\omega(A,B) - \sum_\ell \frac{\lambda_\ell}{2\ell} \mathrm{tr} A^\ell - \sum_k \frac{\mu_k}{2kn^{k-2}} \mathrm{tr} B^k$$

$$K_\omega(A,B) \equiv \left(K_\omega\right)^{abcd}{}_{efgh} A_{abcd} B^{efgh} \quad \left[\left(K_\omega\right)^{abcd}{}_{efgh} \in \mathbb{C}\right]$$

⇩

This has a severe <u>sign problem</u> in Monte Carlo simulations for large N

⇩

We decided to develop a machinery to solve the sign problem.

⇩

Road to the manifold case (2/2)

⇩

We have invented an algorithm,
"tempered, generalized Lefschetz-thimble method (tGLTM)",
which seems to solve the sign problem versatilely:

[MF-Umeda, Alexandru et al. 2016]

⇩

We are now applying the algorithm to the triangle-hinge models.

[A few nice byproducts in this detour]

Our tGLTM actually can be applied to many other fields,
including QCD at finite density, Hubbard model away from half filling, …

[MF-Matsumoto-Umeda 2018, in preparation]

In the course of optimizing the parameters in numerical simulations,
we found that one can introduce a geometry to Markov chain Monte Carlo
which measures the difficulty of transitions between configurations.
This gives the first example of the <u>emergence of geometry in stochastic systems</u>.

[MF-Matsumoto-Umeda 2017, 2018]

⇨ **N. Matsumoto's poster**

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Conclusion and outlook

What we have done:

- We propose a new class of matrix models that generate 3D random volumes.
- The models are characterized by semisimple associative algebras $\mathcal{A}$.
- Although most of the Feynman diagrams are not manifolds,
  one can reduce the possible Feynman diagrams
  to those representing tetrahedral decompositions of 3D manifolds.
- Local matter d.o.f. can be introduced (located on simplices of any dimensions).

Future directions:

- Need to develop analytic treatment. [MF-Sugishita-Umeda (work in progress)]
  (One can actually diagonalize the interaction parts.)
- Developing a machinery to restrict to a particular topology.
- Introducing matter fields corresponding to the target space coords $X^{\mu}(\xi)$
  and investigating the critical behaviors. $\left(\alpha \Rightarrow \mathbf{x} \in \mathbf{R}^d, \quad \lambda_{\alpha\beta} \Rightarrow \lambda_{\mathbf{x},\mathbf{y}}\right)$
- Introducing supersymmetry and investigating the critical behaviors.
- Application to condensed matter physics?

Thank you.