New fermionization formula using only commutation relations.

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・New fermionization method

Examples（１-dim transverse Ising model, Kitaev model,
cluster model, XY model,
2-dim Ising model ）

・Infinite number of solvable models, with c=m/2

・Jordan-Wigner transformation is a special case

・Phase diagram of 2-dim Kitaev model + Wen model

・realizations of the Onsager algebra

Diagonalization of Hamiltonian

partition function

$$Z = \sum_j e^{-\beta E_j}$$

Hamiltonian

$$\begin{aligned} -\beta\mathcal{H} &= \sum_{j=1}^{N} (\text{ spin operators }) \\ &= \sum_p \left(c^\dagger(p)c(p) - \frac{1}{2} \right) \end{aligned}$$

Free fermion

Jordan-Wigner transformation

$$c_k = \exp\left[i\pi \sum_{j=1}^{k-1} s_j^+ s_j^- \right] s_k^-, \qquad c_k^\dagger = \exp\left[i\pi \sum_{j=1}^{k-1} s_j^+ s_j^- \right] s_k^+,$$

c.f.

$$\begin{aligned} e^{\pm i\pi s_j^+ s_j^-} &= 1 + \frac{(\pm i\pi)}{1!} s_j^+ s_j^- + \frac{(\pm i\pi)^2}{2!} s_j^+ s_j^- + \dots \\ &= 1 - 2 s_j^+ s_j^- = -\sigma_j^z. \end{aligned}$$

Fermionization formula

Find the series of operators

$$\eta_1 \quad \eta_2 \quad \eta_3 \quad \eta_4 \quad \eta_5 \quad \cdots$$

which satisfy

$$\eta_j\eta_k = -\eta_k\eta_j \quad (k = j \pm 1 : \text{when adjacent anticommute}), \quad \eta_j^2 = 1 \quad (k = j),$$
$$\eta_j\eta_k = \eta_k\eta_j \quad (|j-k| \geq 2 : \text{otherwise commute}).$$

Then the following Hamiltonian is solvable

$$-\beta\mathcal{H} = \sum_{j=1}^{N} K_j\eta_j$$

with the use of the transformation **we have**

$$\varphi_j = \frac{1}{\sqrt{2}} e^{i\frac{\pi}{2}(j-1)} \eta_0\eta_1\eta_2\cdots\eta_j, \qquad \{\varphi_j, \varphi_k\} = \varphi_j\varphi_k + \varphi_k\varphi_j = \delta_{jk}.$$

Transverse Ising model

$$-\beta\mathcal{H} = K\sum_{j=1}^{N}\sigma_j^z\sigma_{j+1}^z + h\sum_{j=1}^{N}\sigma_j^x$$

that consists of the following operators

$$\sigma_1^x \quad \sigma_1^z\sigma_2^z \quad \sigma_2^x \quad \sigma_2^z\sigma_3^z \quad \sigma_3^x \quad \ldots \quad \sigma_j^z\sigma_{j+1}^z$$

Let $\eta_1 = \sigma_1^x$**,** $\eta_2 = \sigma_1^z\sigma_2^z$**, and generally** $\eta_{2j-1} = \sigma_j^x \quad \eta_{2j} = \sigma_j^z\sigma_{j+1}^z$**, then**

$$\eta_j\eta_k = -\eta_k\eta_j \quad (k = j \pm 1 : \text{when adjacent anticommute}), \quad \eta_j^2 = 1 \quad (k = j),$$
$$\eta_j\eta_k = \eta_k\eta_j \quad (|j-k| \geq 2 : \text{otherwise commute}).$$

The transverse Ising model is, therefore, obtained from

$$\eta_1 \quad \eta_2 \quad \eta_3 \quad \eta_4 \quad \eta_5 \quad \ldots$$

which satisfy the above commutation relation. The Hamiltonian is written as

$$-\beta\mathcal{H} = K\sum_{j=even}^{N}\eta_j + h\sum_{j=odd}^{N}\eta_j$$

Let us introduce the transformation

$$\varphi_j = \eta_1 \eta_2 \cdots \eta_j.$$

If $j \neq k$**,** $\varphi_j \varphi_k = (\eta_1 \cdots \eta_j)(\eta_1 \cdots \eta_j \cdots \eta_k) = (-1)^{j-1} \eta_{j+1} \cdots \eta_k$

$\varphi_k \varphi_j = (\eta_1 \cdots \eta_j \cdots \eta_k)(\eta_1 \cdots \eta_j) = (-1)(-1)^{j-1} \eta_{j+1} \cdots \eta_k = -\varphi_j \varphi_k,$

if $k = j$**,** $\varphi_j \varphi_j = (\eta_1 \cdots \eta_j)(\eta_1 \cdots \eta_j) = (-1)^{j-1}.$

Let us re-define the transformation as

$$\varphi_j = \frac{1}{\sqrt{2}} e^{i \frac{\pi}{2}(j-1)} \eta_1 \eta_2 \cdots \eta_j$$

then we find the commutation relation

$$\varphi_j \varphi_k = -\varphi_k \varphi_j, \quad \varphi_j^2 = \frac{1}{2} \quad \rightarrow \qquad \{\varphi_j, \varphi_k\} = \varphi_j \varphi_k + \varphi_k \varphi_j = \delta_{jk}.$$

New operators φ_j are written explicitely as

$$
\begin{aligned}
\varphi_1 &= \frac{1}{\sqrt{2}}\eta_1 = \frac{1}{\sqrt{2}}\sigma_1^x \\
\varphi_2 &= \frac{1}{\sqrt{2}}e^{i\frac{\pi}{2}}\eta_1\eta_2 = \frac{1}{\sqrt{2}}i\sigma_1^x\sigma_1^z\sigma_2^z = \frac{1}{\sqrt{2}}\sigma_1^y\sigma_2^z \\
\varphi_3 &= \frac{1}{\sqrt{2}}e^{i\frac{\pi}{2}2}\eta_1\eta_2\eta_3 = \frac{1}{\sqrt{2}}i^2\sigma_1^x\sigma_1^z\sigma_2^z\sigma_2^x = \frac{-1}{\sqrt{2}}\sigma_1^y\sigma_2^y \\
\varphi_4 &= \frac{1}{\sqrt{2}}e^{i\frac{\pi}{2}3}\eta_1\eta_2\eta_3\eta_4 = \frac{1}{\sqrt{2}}i^3\sigma_1^x\sigma_1^z\sigma_2^z\sigma_2^x\sigma_2^z\sigma_3^z = \frac{1}{\sqrt{2}}\sigma_1^y\sigma_2^x\sigma_3^z \\
\varphi_5 &= \frac{1}{\sqrt{2}}e^{i\frac{\pi}{2}4}\eta_1\eta_2\cdots\eta_5 = \frac{1}{\sqrt{2}}i^4\sigma_1^x\sigma_1^z\sigma_2^z\sigma_2^x\sigma_2^z\sigma_3^z\sigma_3^x = \frac{-1}{\sqrt{2}}\sigma_1^y\sigma_2^x\sigma_3^y
\end{aligned}
$$

Introducing an initial operator $\eta_0 = i\sigma_1^z$ (to avoid irregular behaviors coming from the boundary) we find

$$
\begin{aligned}
\varphi_{2j} &= \frac{1}{\sqrt{2}}e^{i\frac{\pi}{2}(2j-1)}\eta_0\eta_1\eta_2\cdots\eta_{2j} = \frac{1}{\sqrt{2}}\sigma_1^x\sigma_2^x\sigma_3^x\cdots\sigma_j^x\sigma_{j+1}^z \\
\varphi_{2j+1} &= \frac{1}{\sqrt{2}}e^{i\frac{\pi}{2}2j}\eta_0\eta_1\eta_2\cdots\eta_{2j+1} = \frac{-1}{\sqrt{2}}\sigma_1^x\sigma_2^x\sigma_3^x\cdots\sigma_j^x\sigma_{j+1}^y
\end{aligned}
$$

Hamiltonian is written by two-body terms of φ_j

$$-\beta\mathcal{H} = K\sum_{j=\mathrm{odd}}^{N}(-2i)\varphi_j\varphi_{j+1} + h\sum_{j=\mathrm{even}}^{N}(-2i)\varphi_j\varphi_{j+1},$$

this is because η_{j+1} is proportional to $\varphi_j\varphi_{j+1}$

$$\begin{aligned}\varphi_j\varphi_{j+1} &= \frac{1}{\sqrt{2}}e^{i\frac{\pi}{2}(j-1)}\eta_0\eta_1\eta_2\cdots\eta_j\frac{1}{\sqrt{2}}e^{i\frac{\pi}{2}j}\eta_0\eta_1\eta_2\cdots\eta_j\eta_{j+1} \\ &= (\frac{1}{\sqrt{2}})^2 e^{i\frac{\pi}{2}}\eta_{j+1} = \frac{i}{2}\eta_{j+1}.\end{aligned}$$

Boundary term

$$\begin{aligned}\varphi_{2N-1}\varphi_0 &= \frac{1}{\sqrt{2}}e^{i\frac{\pi}{2}(2N-2)}\eta_0\eta_1\cdots\eta_{2N}\frac{1}{\sqrt{2}}\eta_0\eta_1 \\ &= \frac{i}{2}(-\sigma_1^x\cdots\sigma_N^x)\sigma_N^z\sigma_1^z\end{aligned}$$

The cyclic boundary condition can be introduced

$$\begin{aligned}\sigma_1^x\cdots\sigma_N^x = -1 \quad &\rightarrow \quad \varphi_{2N} = +\varphi_0 \\ \sigma_1^x\cdots\sigma_N^x = +1 \quad &\rightarrow \quad \varphi_{2N} = -\varphi_0\end{aligned}$$

Fourier transformation

$$\varphi_j = \frac{1}{\sqrt{2N}} \sum_{0<q<\pi} (e^{iqj}C(q) + e^{-iqj}C^\dagger(q)),$$

where $C(q)$ are the fermion operators

$$\{C^\dagger(p), C(q)\} = \delta_{pq}, \quad \{C^\dagger(p), C^\dagger(q)\} = \{C(p), C(q)\} = 0.$$

Finally, the Hamiltonian is written by two-body terms of $C(q)$ and hence diagonaizable

$$\begin{aligned} -\beta\mathcal{H} &= K\sum_{j=1}^{N/2}(-2i)\varphi_{2j-2}\varphi_{2j-1} + h\sum_{j=1}^{N/2}(-2i)\varphi_{2j-1}\varphi_{2j} \\ &= (-i)\sum_{0<q<\pi}[(K-h)e^{-iq}C(q)C(\pi-q) + (K-h)e^{iq}C^\dagger(q)C^\dagger(\pi-q) \\ &\qquad (K+h)e^{-iq}C(q)C^\dagger(q) + (K+h)e^{iq}C^\dagger(q)C(q)] \end{aligned}$$

Introducing the basis states $|0\,0\rangle$, $|1\,0\rangle = C^\dagger(q)|0\,0\rangle$, $|0\,1\rangle = C^\dagger(\pi - q)|0\,0\rangle$, $|1\,1\rangle = C^\dagger(\pi - q)C^\dagger(q)|0\,0\rangle$, the Hamiltonian can be expressed as

$$-\beta\mathcal{H} = 2\sum_{0<q<\pi/2}\begin{bmatrix} (K+h)\sin q & 0 & 0 & i(K-h)\cos q \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i(K-h)\cos q & 0 & 0 & -(K+h)\sin q \end{bmatrix}$$

Eigenvalues

$$0,\ \ 0,\ \ \pm\Lambda_q, \qquad \Lambda_q = 2\sqrt{K^2 + h^2 - 2Kh\cos 2q}$$

Partition function

$$Z = \prod_{0<q<\pi/2}(1 + 1 + e^{\Lambda_q} + e^{-\Lambda_q}) = \prod_{0<q<\pi/2}(e^{\frac{1}{2}\Lambda_q} + e^{-\frac{1}{2}\Lambda_q})^2$$

The free energy

$$-\beta f = \lim_{N\to\infty}\frac{\log Z}{N} = \frac{1}{\pi}\int_0^{\pi/2}\log(e^{\frac{1}{2}\Lambda_q} + e^{-\frac{1}{2}\Lambda_q})\,dq.$$

Summary of the formula

Find the series of operators

$$\eta_1 \quad \eta_2 \quad \eta_3 \quad \eta_4 \quad \eta_5 \quad \cdots$$

which satisfy

$$\eta_j \eta_k = -\eta_k \eta_j \quad (k = j \pm 1 : \text{when adjacent anticommute}), \quad \eta_j^2 = 1 \quad (k = j),$$
$$\eta_j \eta_k = \eta_k \eta_j \quad (|j-k| \geq 2 : \text{otherwise commute}).$$

Then the following Hamiltonian is solvable

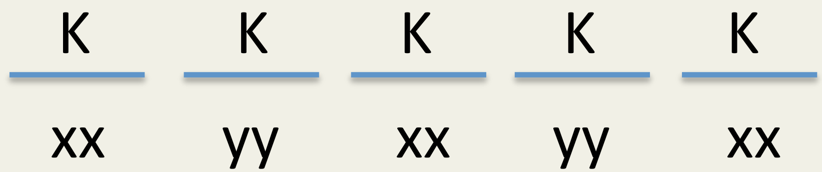
$$-\beta \mathcal{H} = \sum_{j=1}^{N} K_j \eta_j$$

with the use of the transformation

$$\varphi_j = \frac{1}{\sqrt{2}} e^{i\frac{\pi}{2}(j-1)} \eta_0 \eta_1 \eta_2 \cdots \eta_j.$$

One-dimensional quantum systems

Period=1



$$-\beta\mathcal{H} = K\Big(\sum_{j=\text{odd}}^{N} \sigma_j^x \sigma_{j+1}^x + \sum_{j=\text{even}}^{N} \sigma_j^y \sigma_{j+1}^y\Big)$$

Series of operators

$$\eta_1 = \sigma_1^x \sigma_2^x \quad \eta_2 = \sigma_2^y \sigma_3^y \quad \eta_3 = \sigma_3^x \sigma_4^x \quad \eta_4 = \sigma_4^y \sigma_5^y \quad \dots \quad \eta_{2N} = \sigma_N^y \sigma_1^y$$

Hamiltonian

$$-\beta\mathcal{H} = K\sum_{j=1}^{N} \eta_j$$

Fourier transformation

$$\varphi_j = \frac{1}{\sqrt{2N}} \sum_{0<q<\pi} \left(e^{iqj} C(q) + e^{-iqj} C^\dagger(q)\right)$$

where $\{C^\dagger(p), C(q)\} = \delta_{pq}, \quad \{C^\dagger(p), C^\dagger(q)\} = \{C(p), C(q)\} = 0.$

Partition function $\quad Z = \prod_{0<q<\pi} 2(e^{\frac{1}{2}\Lambda_q} + e^{-\frac{1}{2}\Lambda_q})^2 \qquad \Lambda_q = 4K\sin q$

Free energy $\quad -\beta f = \frac{1}{\pi}\int_0^\pi \log(e^{\frac{1}{2}\Lambda_q} + e^{-\frac{1}{2}\Lambda_q})\, dq + \frac{1}{2}\log 2$

Period=2

K1	K2	K1	K2	K1
xx	yy	xx	yy	xx

$$-\beta\mathcal{H} = K_1 \sum_{j=\mathrm{odd}}^{N} \sigma_j^x \sigma_{j+1}^x + K_2 \sum_{j=\mathrm{even}}^{N} \sigma_j^y \sigma_{j+1}^y$$

1-dim. Kitaev model

Operators

$$\eta_1 = \sigma_1^x \sigma_2^x \quad \eta_2 = \sigma_2^y \sigma_3^y \quad \eta_3 = \sigma_3^x \sigma_4^x \quad \eta_4 = \sigma_4^y \sigma_5^y \quad \dots \quad \eta_{2N} = \sigma_N^y \sigma_1^y$$

then the Hamiltonian is written as

Change of notation

$$-\beta\mathcal{H} = K_1 \sum_{j=\mathrm{odd}}^{N} \eta_j + K_2 \sum_{j=\mathrm{even}}^{N} \eta_j$$

Fourier transformation, let $\varphi_1 = \varphi_1(1)$, $\varphi_2 = \varphi_2(1)$, $\varphi_3 = \varphi_1(2)$, $\varphi_4 = \varphi_2(2)$

$$\varphi_k(j) = \frac{1}{\sqrt{N}} \sum_{0<q<\pi} (e^{iqj} C_k(q) + e^{-iqj} C_k^\dagger(q)) \qquad k = 1, 2$$

Partition function $\quad Z = \prod_{0<q<\pi} (e^{\frac{1}{2}\Lambda_q} + e^{-\frac{1}{2}\Lambda_q})^2 \quad \Lambda_q = 2\sqrt{K_1^2 + K_2^2 + 2K_1K_2\cos q}$

The free energy is identical to that of the transverse Ising model

$$-\beta\mathcal{H} = K \sum_{j=1}^{N} \sigma_j^z \sigma_{j+1}^z + h \sum_{j=1}^{N} \sigma_j^x$$

K1 K2 K3 K1 K2

Period=3

Series of operators

$$\varphi_1(1),\ \ \varphi_2(1),\ \ \varphi_3(1),\ \ \varphi_1(2),\ \ \varphi_2(2),\ \ \varphi_3(2),\ \ \varphi_1(3),\ \ \ldots$$

Hamiltonian

$$\begin{aligned} -\beta\mathcal{H} &= K_1\sum_{j=1}^{N}\varphi_1(j)\varphi_2(j) + K_2\sum_{j=1}^{N}\varphi_2(j)\varphi_3(j) + K_3\sum_{j=1}^{N}\varphi_3(j)\varphi_1(j+1) \\ &= K_1(-2i)\sum_{0<q<\pi}[C_1^\dagger(q)C_2(q) + C_1(q)C_2^\dagger(q)] \\ &+ K_2(-2i)\sum_{0<q<\pi}[C_2^\dagger(q)C_3(q) + C_2(q)C_3^\dagger(q)] \\ &+ K_3(-2i)\sum_{0<q<\pi}[e^{iq}C_3^\dagger(q)C_1(q) + e^{-iq}C_3(q)C_1^\dagger(q)] \end{aligned}$$

Fourier transformation

$$\varphi_k(j) = \frac{1}{\sqrt{N}}\sum_{0<q<\pi}(e^{iqj}C_k(q) + e^{-iqj}C_k^\dagger(q)), \quad k = 1,2,3$$

Partitiom function

$$Z = \prod_{0<q<\pi}\sum_{l=1}^{8} e^{\lambda_l}$$

the eigenvalues are obtained from the following equation

$$0 = \lambda^3 - 4(K_1^2 + K_2^2 + K_3^2)\lambda \pm 16K_1K_2K_3\sin q$$

when $K_1 = K_2 = K_3 \ (= K)$

$$\begin{aligned} 0 &= \lambda^3 - 12K^2\lambda + 16K^3 \sin q, \qquad \sin q = -4 \sin\frac{q+\pi}{3} \sin\frac{q+2\pi}{3} \sin\frac{q+3\pi}{3} \\ &= (\lambda - 4K\sin\frac{q+\pi}{3})(\lambda - 4K\sin\frac{q+5\pi}{3})(\lambda - 4K\sin\frac{q+3\pi}{3}) \end{aligned}$$

after all λ_l **are obtained as**

$$\lambda = 0,\ \ 0,\ \ 4K\sin\frac{q+n\pi}{3} \qquad n = 0,1,2,3,4,5$$

Partition function

$$\begin{aligned} Z &= \prod_{0<q<\pi} \left[1 + 1 + \sum_{n=0}^{5} e^{4K\sin\frac{q+n\pi}{3}}\right] \quad q = \frac{l}{N}\pi \\ &= \prod_{0<p<\pi} \left[e^{2K\sin\frac{p}{3}} + e^{-2K\sin\frac{p}{3}}\right] \qquad p = \frac{l}{3N}\pi \end{aligned}$$

Period=4

K1 K2 K3 K4 K1

Series of operators

$$\varphi_1(1),\ \ \varphi_2(1),\ \ \varphi_3(1),\ \ \varphi_4(1),\ \ \varphi_1(2),\ \ \varphi_2(2),\ \ \ldots$$

Hamiltonian

$$\begin{aligned}
-\beta\mathcal{H} &= K_1\sum_{j=1}^{N}\varphi_1(j)\varphi_2(j) + K_2\sum_{j=1}^{N}\varphi_2(j)\varphi_3(j) + K_3\sum_{j=1}^{N}\varphi_3(j)\varphi_4(j) + K_4\sum_{j=1}^{N}\varphi_4(j)\varphi_1(j+1) \\
&= K_1(-2i)\sum_{0<q<\pi}[C_1^\dagger(q)C_2(q) + C_1(q)C_2^\dagger(q)] + K_2(-2i)\sum_{0<q<\pi}[C_2^\dagger(q)C_3(q) + C_2(q)C_3^\dagger(q)] \\
&+ K_3(-2i)\sum_{0<q<\pi}[C_3^\dagger(q)C_4(q) + C_3(q)C_4^\dagger(q)] \\
&+ K_4(-2i)\sum_{0<q<\pi}[e^{iq}C_4^\dagger(q)C_1(q) + e^{-iq}C_4(q)C_1^\dagger(q)]
\end{aligned}$$

Partition function

$$Z = \prod_{0<q<\pi}\sum_{l=1}^{16} e^{\lambda_l}$$

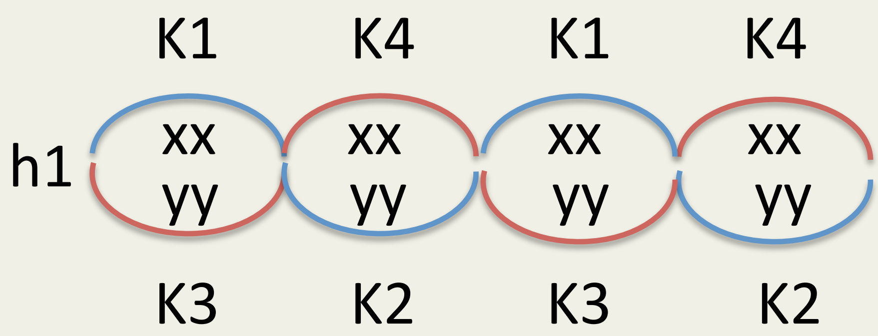
eigenvalues λ_l are $0, 0, 0, 0$ and the solutions of

$$0 = \lambda^4 - 4(K_1^2 + K_2^2 + K_3^2 + K_4^2)\lambda^2 + 16(K_1^2K_3^2 + K_2^2K_4^2 - 2K_1K_2K_3K_4\cos q)$$

are 2-fold degenerated eigenvalues, and the solutions of

$$\begin{aligned}
0 &= \lambda^4 - 8(K_1^2 + K_2^2 + K_3^2 + K_4^2)\lambda^2 \\
&+ 16(K_1^2 + K_2^2 + K_3^2 + K_4^2)^2 - 64(K_1^2K_3^2 + K_2^2K_4^2 - 2K_1K_2K_3K_4\cos q)
\end{aligned}$$

1-dim. XY model



two series

$$
\begin{aligned}
\varphi_1(j) &= \frac{1}{\sqrt{2}}\sigma_1^z\sigma_2^z\cdots\sigma_{2j-2}^z\sigma_{2j-1}^y \quad \varphi_2(j) = \frac{1}{\sqrt{2}}\sigma_1^z\sigma_2^z\cdots\sigma_{2j-1}^z\sigma_{2j}^x \\
\varphi_3(j) &= \frac{1}{\sqrt{2}}\sigma_1^z\sigma_2^z\cdots\sigma_{2j-2}^z\sigma_{2j-1}^x \quad \varphi_4(j) = \frac{1}{\sqrt{2}}\sigma_1^z\sigma_2^z\cdots\sigma_{2j-1}^z\sigma_{2j}^y
\end{aligned}
$$

then

external field

$$
\varphi_1(j)\varphi_2(j) = \frac{i}{2}\sigma_{2j-1}^x\sigma_{2j}^x \quad \varphi_2(j)\varphi_1(j+1) = \frac{-i}{2}\sigma_{2j}^y\sigma_{2j+1}^y \quad \varphi_1(j)\varphi_3(j) = \frac{-i}{2}\sigma_{2j-1}^z
$$

$$
\varphi_3(j)\varphi_4 j) = \frac{-i}{2}\sigma_{2j-1}^y\sigma_{2j}^y \quad \varphi_4(j)\varphi_3(j+1) = \frac{i}{2}\sigma_{2j}^x\sigma_{2j+1}^x \quad \varphi_2(j)\varphi_4(j) = \frac{i}{2}\sigma_{2j}^z
$$

Hamiltonian

$$
\begin{aligned}
-\beta\mathcal{H} &= K_1\sum_{j=1}^N \varphi_1(j)\varphi_2(j) + K_2\sum_{j=1}^N \varphi_2(j)\varphi_3(j+1) + h_1\sum_{j=1}^N \varphi_1(j)\varphi_3(j) \\
&+ K_3\sum_{j=1}^N \varphi_3(j)\varphi_4(j) + K_4\sum_{j=1}^N \varphi_4(j)\varphi_3(j+1) + h_2\sum_{j=1}^N \varphi_2(j)\varphi_4(j)
\end{aligned}
$$

Partition function

$$Z = \prod_{0<q<\pi} \sum_{l=1}^{16} e^{\lambda_l} = \prod_{0<q<\pi} (e^P + e^{-P} + e^Q + e^{-Q})^2$$

$$P = \Big(A + 2|B|\Big)^{1/2} \qquad Q = \Big(A - 2|B|\Big)^{1/2}$$

$$\begin{aligned} A &= K_1^2 + K_2^2 + K_3^2 + K_4^2 + 2K_1K_2\cos q + 2K_3K_4\cos q + h_1^2 + h_2^2 \\ B &= (K_1 + K_2e^{-iq})(K_3 + K_4e^{iq}) - h_1h_2 \end{aligned}$$

Let $K_1 = K_4 = K(1+\gamma)$**,** $K_3 = K_2 = K(1-\gamma)$**,** $h_1 = h_2 = h$**, then**

$$\begin{aligned} Z &= \prod_{0<p<\pi} (e^R + e^{-R})^2 \\ R &= \sqrt{(h - 2K\cos p)^2 + (2K\gamma\sin p)^2} \end{aligned}$$

(Lieb-Schultz-Mattis 1961, Katsura 1962, Niemeijer 1967)

Relations with other methods

Transformation by Nambu

Nambu solved the equare lattice Ising model (Nambu 1950)
The inverse of his transformation is our transformation for the XY chain

$$\varphi_1(j) = \frac{(-1)^{j-1}}{\sqrt{2}}(\prod_{k=1}^{2j-2}\sigma_k^z)\sigma_{2j-1}^y, \quad \varphi_2(j) = \frac{(-1)^{j-1}}{\sqrt{2}}(\prod_{k=1}^{2j-1}\sigma_k^z)\sigma_{2j}^x,$$

$$\varphi_3(j) = \frac{(-1)^{j-1}}{\sqrt{2}}(\prod_{k=1}^{2j-2}\sigma_k^z)\sigma_{2j-1}^x, \quad \varphi_4(j) = \frac{(-1)^{j}}{\sqrt{2}}(\prod_{k=1}^{2j-1}\sigma_k^z)\sigma_{2j}^y,$$

i) with the initial operators $\eta_0 = -ix_2$ **and** $\zeta_0 = -ix_1$
ii) the sign of the additional phase is opposite $\exp(-i\frac{\pi}{2}(j-1))$

The Jordan-Wigner transformation is a special case

The Jordan-Wigner transformation is a special case of our formula.
Let

$$\Psi_j = u_3\varphi_3(j) + u_1\varphi_1(j), \qquad \Psi_j^\dagger = u_3^*\varphi_3(j) + u_1^*\varphi_1(j),$$

where $\varphi_3(j)$ and $\varphi_1(j)$ were introduced in the case of the XY model.
Canonical condition

$$\begin{aligned} \delta_{jk} &= \{\Psi_j^\dagger, \Psi_k\} = (u_3^* u_3 + u_1^* u_1)\delta_{jk}, & u_3 &= \frac{1}{\sqrt{2}}e^{i\theta}, \\ 0 &= \{\Psi_j, \Psi_k\} = (u_3 u_3 + u_1 u_1)\delta_{jk}, & u_1 &= \frac{\pm i}{\sqrt{2}}e^{i\theta}. \end{aligned}$$

Let $\theta = (j-1)\pi$ and choose negative sign, then

$$\Psi_j = \frac{(-1)^{j-1}}{\sqrt{2}}\Big(\varphi_3(j) - i\varphi_1(j)\Big), \quad \Psi_j^\dagger = \frac{(-1)^{j-1}}{\sqrt{2}}\Big(\varphi_3(j) + i\varphi_1(j)\Big).$$

$$= \exp\Big[i\pi\sum_{k=1}^{2j-2} s_k^+ s_k^-\Big] s_{2j-1}^-, \qquad = \exp\Big[-i\pi\sum_{k=1}^{2j-2} s_k^+ s_k^-\Big] s_{2j-1}^+.$$

These are the operators $c_{2j-1}$and $c_{2j-1}^\dagger$ in the Jordan-Wigner transformation.
Similarly we can obtain c_{2j} and $c_{2j}^\dagger$ from $\varphi_4(j)$ and $\varphi_2(j)$.

Infinite series of solvable spin chains

1-dim. cluster model

$$-\beta\mathcal{H} = K_1\sum_{j=1}^{N}\sigma_j^x\sigma_{j+1}^z\sigma_{j+2}^x + (K_2\sum_{j=1}^{N}\sigma_j^y\sigma_{j+1}^z\sigma_{j+2}^y + K_3\sum_{j=1}^{N}\sigma_j^y\sigma_{j+1}^y)$$

Generalized cluster model

$$-\beta\mathcal{H} = K_1\sum_{j=1}^{N}\sigma_j^x\sigma_{j+1}^z\sigma_{j+2}^x + K_2\sum_{j=1}^{N}\sigma_j^x\,\mathbf{1}_{j+1}\sigma_{j+2}^x$$

Operators $\sigma_1^x\sigma_2^z\sigma_3^x$, $\sigma_2^x\,\mathbf{1}_3\sigma_4^x$, $\sigma_3^x\sigma_4^z\sigma_5^x$, $\sigma_4^x\,\mathbf{1}_5\sigma_6^x$, $\ldots$ satisfy the condition, and hence can be solved.

This model is diagonalized by the transformation

$$\varphi_{2j-1} = \frac{1}{\sqrt{2}}(\,\mathbf{1}_1\,\sigma_2^z\,\mathbf{1}_3\,\sigma_4^z\,\mathbf{1}_5\,\sigma_6^z\cdots\cdots\,\mathbf{1}_{2j-1})\,\sigma_{2j}^y\,\sigma_{2j+1}^x,$$

$$\varphi_{2j} = \frac{1}{\sqrt{2}}(\,\mathbf{1}_1\,\sigma_2^z\,\mathbf{1}_3\,\sigma_4^z\,\mathbf{1}_5\,\sigma_6^z\cdots\cdots\,\mathbf{1}_{2j-1}\,\sigma_{2j}^z)\sigma_{2j+1}^x\sigma_{2j+2}^x,$$

which is different from the Jordan-Wigner transformation

Cannot be solved by the Jordan-Wigner transformation

Solvable series of spin chains (Minami, Nuclear Physics B 2017)

Table 1: Examples of solvable Hamiltonians $-\beta\mathcal{H}(k,n,l)$ obtained as a linear combination of η_j and their shifted operators. Generalizations following (7) of these Hamiltonians can be found in Table 2.

(k,n,l)	$-\beta\mathcal{H}(k,n,l)$	
$(1,n,l)$	$K_1\sum_{j=1}^{N}\left(\prod_{\nu=1}^{n}\sigma^x_{j+\nu-1}\right)\left(\prod_{\nu=1}^{l}\sigma^z_{j+n+\nu-1}\right)\left(\prod_{\nu=1}^{n}\sigma^x_{j+n+l+\nu-1}\right)$	$+K_2\sum_{j=1}^{N}\sigma^x_j\sigma^x_{j+1}$
$(2,n,l)$	$K_1\sum_{j=1}^{N}\left(\prod_{\nu=1}^{n}\sigma^x_{j+\nu-1}\right)\left(\prod_{\nu=1}^{l}\sigma^z_{j+n+\nu-1}\right)\left(\prod_{\nu=1}^{n}\sigma^x_{j+n+l+\nu-1}\right)$	$+K_2\sum_{j=1}^{N}\sigma^y_j\sigma^y_{j+1}$
$(3,n,l)$	$K_1\sum_{j=1}^{N}\left(\prod_{\nu=1}^{n}\sigma^x_{j+\nu-1}\right)\sigma^z_{j+n}\left(\prod_{\nu=1}^{n}\sigma^x_{j+n+\nu}\right)$	$+K_2\sum_{j=1}^{N}\sigma^x_j\left(\prod_{\nu=1}^{l}\mathbf{1}_{j+\nu}\right)\sigma^x_{j+l+1}$
$(4,n,-)$	$K_1\sum_{j=1}^{N}\sigma^x_j\left(\prod_{\nu=1}^{n}\sigma^z_{j+\nu}\right)\sigma^x_{j+n+1}$	$+K_2\sum_{j=1}^{N}\sigma^z_j\left(\prod_{\nu=1}^{n}\mathbf{1}_{j+\nu}\right)\sigma^z_{j+n+1}$
$(5,n,-)$	$K_1\sum_{j=1}^{N}\left(\prod_{\nu=1}^{n}\sigma^z_{j+\nu-1}\right)$	$+K_2\sum_{j=1}^{N}\sigma^x_j\sigma^x_{j+1}$
$(6,n,-)$	$K_1\sum_{j=1}^{N}\sigma^x_j\left(\prod_{\nu=1}^{n}\mathbf{1}_{j+\nu}\right)\sigma^x_{j+n+1}$	$+K_2\sum_{j=1}^{N}\sigma^z_j$
$(7,n,l)$	$K_1\sum_{j=1}^{N}\sigma^x_j\left(\prod_{\nu=1}^{n}\sigma^z_{j+\nu}\right)\sigma^x_{j+n+1}$	$+K_2\sum_{j=1}^{N}\sigma^x_j\left(\prod_{\nu=1}^{l}\sigma^z_{j+\nu}\right)\sigma^x_{j+l+1}$
$(8,n,l)$	$K_1\sum_{j=1}^{N}\sigma^x_j\left(\prod_{\nu=1}^{n}\sigma^z_{j+\nu}\right)\sigma^x_{j+n+1}$	$+K_2\sum_{j=1}^{N}\sigma^y_j\left(\prod_{\nu=1}^{l}\sigma^z_{j+\nu}\right)\sigma^y_{j+l+1}$
$(9,n,-)$	$K_1\sum_{j=1}^{N}\sigma^x_j\sigma^x_{j+1}\sigma^x_{j+2}\left(\prod_{\nu=1}^{n}\sigma^z_{j+2+\nu}\right)\sigma^x_{j+n+3}\sigma^x_{j+n+4}\sigma^x_{j+n+5}$	$+K_2\sum_{j=1}^{N}\sigma^x_j\left(\prod_{\nu=1}^{n+2}\sigma^z_{j+\nu}\right)\sigma^x_{j+n+3}$
$(10,n,-)$	$K_1\sum_{j=1}^{N}\sigma^x_j\sigma^x_{j+1}\left(\prod_{\nu=1}^{n}\sigma^z_{j+1+\nu}\right)\sigma^x_{j+n+2}\sigma^x_{j+n+3}$	$+K_2\sum_{j=1}^{N}\left(\prod_{\nu=1}^{n+2}\sigma^z_{j+\nu-1}\right)$
$(11,-,-)$	$K_1\sum_{j=1}^{N}\sigma^x_j\sigma^x_{j+1}\sigma^z_{j+2}\sigma^x_{j+3}\sigma^x_{j+4}$	$+K_2\sum_{j=1}^{N}\sigma^x_j\sigma^z_{j+1}\sigma^x_{j+2}$

Solvable series of spin chains

Generally the following Hamiltonians can be diagonalized

$$-\beta\mathcal{H} = \sum_l K_l \sum_{j=1}^{N} (-2i)\varphi_p(j)\varphi_q(j+l) \quad p,\ q = 1 \text{ or } 2$$

Two-body terms of C(q)

For example

$$\begin{aligned}-\beta\mathcal{H} &= \sum_{j=1}^{N} [\ \cdots + K_{-1}(-2i)\varphi_2(j)\varphi_1(j-1) + K_0(-2i)\varphi_2(j)\varphi_1(j) \\ &\qquad + K_1(-2i)\varphi_2(j)\varphi_1(j+1) + K_2(-2i)\varphi_2(j)\varphi_1(j+2) + \cdots] \\ &= \sum_{j=1}^{N} [\ \cdots + K_{-1}\eta_{2j-3}\eta_{2j-2}\eta_{2j-1} + \underline{K_0\eta_{2j-1} + K_1\eta_{2j}} + K_2\eta_{2j}\eta_{2j+1}\eta_{2j+2} + \cdots].\end{aligned}$$

In the case of operators $\eta_{2j-1} = \sigma_j^z$ **and** $\eta_{2j} = \sigma_j^x\sigma_{j+1}^x$

$$\begin{aligned}-\beta\mathcal{H} &= \sum_{j=1}^{N} [\ \cdots + K_{-1}\sigma_{j-1}^y\sigma_j^y + \underline{K_0\sigma_j^z + K_1\sigma_j^x\sigma_{j+1}^x} + K_2\sigma_j^x\sigma_{j+1}^z\sigma_{j+2}^x + \cdots] \\ &= \sum_{j=1}^{N} [\ \cdots + \quad YY \quad + \quad Z \quad + \quad XX \quad + \quad XZX \quad + \cdots]\end{aligned}$$

Transverse Ising

(Suzuki 1971)

Table 2: Each (k,n,l) provides a solvable Hamiltonian, where six interactions in (??) are explicitly written. One can find the operators $\{\eta_j^{(k,n,l)}\}$ in the second and third row of the first column, i.e. $\eta_{2j-1}^{(k,n,l)} = +2i\varphi_2^{(k,n,l)}(j)\varphi_1^{(k,n,l)}(j)$ and $\eta_{2j}^{(k,n,l)} = -2i\varphi_2^{(k,n,l)}(j)\varphi_1^{(k,n,l)}(j+1)$, from which a solvable series of interactions are generated. The initial operator $\eta_0^{(k,n,l)}$ and the transformation $\varphi_j^{(k,n,l)}$ that diagonalize the Hamiltonian are given in the last row. The first case $(k,n,l)=(1,0,1)$ includes the transverse Ising model, the XY model, and the cluster model, as special cases. The second case $(k,n,l)=(2,1,0)$ includes the one-dimensional Kitaev model. The operators $\sigma_j^x\sigma_{j+1}^z\sigma_{j+2}^x$ are the stabilizers of the cluster state. Models except the cases $(1,0,1)$ and $(2,1,0)$ cannot be solved by the Jordan-Wigner transformation.

(k,n,l)		
$-2i\varphi_2^{(k,n,l)}(j)\varphi_1^{(k,n,l)}(j-1)$		
$+2i\varphi_2^{(k,n,l)}(j)\varphi_1^{(k)}(j)$	$=\eta_{2j-1}^{(k,n,l)}$	$-2i\varphi_1^{(k,n,l)}(j)\varphi_1^{(k,n,l)}(j+1)$
$-2i\varphi_2^{(k,n,l)}(j)\varphi_1^{(k,n,l)}(j+1)$	$=\eta_{2j}^{(k,n,l)}$	$-2i\varphi_2^{(k,n,l)}(j)\varphi_2^{(k,n,l)}(j+1)$
$+2i\varphi_2^{(k,n,l)}(j)\varphi_1^{(k,n,l)}(j+2)$		
$\eta_0^{(k,n,l)}$		$\varphi_{2j}^{(k,n,l)}$ and $\varphi_{2j+1}^{(k,n,l)}$ $(j=0,1,2,3,\ldots)$

Transv. Ising model
XY model
cluster model

Jordan-Wigner trans.

$(1,0,1)$		
$\sigma_{j-1}^y\sigma_j^y$		
σ_j^z	$\sigma_j^y\sigma_{j+1}^x$	
$\sigma_j^x\sigma_{j+1}^x$	$\sigma_j^x\sigma_{j+1}^y$	
$\sigma_j^x\sigma_{j+1}^z\sigma_{j+2}^x$		
$i\sigma_1^x$	$\varphi_{2j}^{(1,0,1)} = \frac{1}{\sqrt{2}}(\prod_{\nu=1}^{j}\sigma_\nu^z)\sigma_{j+1}^x$	$\varphi_{2j+1}^{(1,0,1)} = \frac{1}{\sqrt{2}}(\prod_{\nu=1}^{j}\sigma_\nu^z)\sigma_{j+1}^y$

XY model

Jordan-Wigner trans.

$(2,1,0)$		
$\sigma_{2j-3}^x\sigma_{2j-2}^z\sigma_{2j-1}^z\sigma_{2j}^x$		
$\sigma_{2j-1}^x\sigma_{2j}^x$	$\sigma_{2j-1}^x\sigma_{2j}^z\sigma_{2j+1}^y$	
$\sigma_{2j}^y\sigma_{2j+1}^y$	$\sigma_{2j}^y\sigma_{2j+1}^z\sigma_{2j+2}^x$	
$\sigma_{2j}^y\sigma_{2j+1}^z\sigma_{2j+2}^z\sigma_{2j+3}^y$		
$i\sigma_1^y$	$\varphi_{2j}^{(2,1,0)} = \frac{(-1)^j}{\sqrt{2}}(\prod_{\nu=1}^{2j}\sigma_\nu^z)\sigma_{2j+1}^y$	$\varphi_{2j+1}^{(2,1,0)} = \frac{(-1)^j}{\sqrt{2}}(\prod_{\nu=1}^{2j+1}\sigma_\nu^z)\sigma_{2j+2}^x$

Cannot be solved by the Jordan-Wigner transformation

X Z X + X 1 X

(3, 1, 1)	
$\sigma^x_{2j-3}\sigma^y_{2j-2}1_{2j-1}\sigma^y_{2j}\sigma^x_{2j+1}$	
$\sigma^x_{2j-1}\sigma^z_{2j}\sigma^x_{2j+1}$	$\sigma^x_{2j-1}\sigma^y_{2j}\sigma^x_{2j+1}\sigma^x_{2j+2}$
$\sigma^x_{2j}1_{2j+1}\sigma^x_{2j+2}$	$\sigma^x_{2j}\sigma^x_{2j+1}\sigma^y_{2j+2}\sigma^x_{2j+3}$
$\sigma^x_{2j}\sigma^x_{2j+1}\sigma^z_{2j+2}\sigma^x_{2j+3}\sigma^x_{2j+4}$	
$i\sigma^x_1\sigma^x_2$	$\varphi^{(3,1,1)}_{2j} = \frac{1}{\sqrt{2}}(\prod_{\nu=1}^{j} 1_{2\nu-1}\sigma^z_{2\nu})\sigma^x_{2j+1}\sigma^x_{2j+2}$
	$\varphi^{(3,1,1)}_{2j+1} = \frac{1}{\sqrt{2}}(\prod_{\nu=1}^{j} 1_{2\nu-1}\sigma^z_{2\nu})1_{2j+1}\sigma^y_{2j+2}\sigma^x_{2j+3}$

X Z X + Z 1 Z

(3, 1, 2)	
$\sigma^x_{3j-5}\sigma^y_{3j-4}\sigma^x_{3j-3}\sigma^x_{3j-2}\sigma^y_{3j-1}\sigma^x_{3j}$	
$\sigma^x_{3j-2}\sigma^z_{3j-1}\sigma^x_{3j}$	$\sigma^x_{3j-2}\sigma^y_{3j-1}\sigma^x_{3j}1_{3j+1}\sigma^x_{3j+2}$
$\sigma^x_{3j-1}1_{3j}1_{3j+1}\sigma^x_{3j+2}$	$\sigma^x_{3j-1}1_{3j}\sigma^x_{3j+1}\sigma^y_{3j+2}\sigma^x_{3j+3}$
$\sigma^x_{3j-1}1_{3j}\sigma^x_{3j+1}\sigma^z_{3j+2}\sigma^x_{3j+3}1_{3j+4}\sigma^x_{3j+5}$	
$i1_1\sigma^x_2 1_3$	$\varphi^{(3,1,2)}_{2j} = \frac{1}{\sqrt{2}}(\prod_{\nu=1}^{j} \sigma^x_{3\nu-2}\sigma^z_{3\nu-1}\sigma^x_{3\nu})1_{3j+1}\sigma^x_{3j+2}$
	$\varphi^{(3,1,2)}_{2j+1} = \frac{1}{\sqrt{2}}(\prod_{\nu=1}^{j} \sigma^x_{3\nu-2}\sigma^z_{3\nu-1}\sigma^x_{3\nu})\sigma^x_{3j+1}\sigma^y_{3j+2}\sigma^x_{3j+3}$

(3, 1, 3)	
$\sigma^x_{4j-7}\sigma^y_{4j-6}\sigma^x_{4j-5}1_{4j-4}\sigma^x_{4j-3}\sigma^y_{4j-2}\sigma^x_{4j-1}$	
$\sigma^x_{4j-3}\sigma^z_{4j-2}\sigma^x_{4j-1}$	$\sigma^x_{4j-3}\sigma^y_{4j-2}\sigma^x_{4j-1}1_{4j}1_{4j+1}\sigma^x_{4j+2}$
$\sigma^x_{4j-2}1_{4j-1}1_{4j}1_{4j+1}\sigma^x_{4j+2}$	$\sigma^x_{4j-2}1_{4j-1}1_{4j}\sigma^x_{4j+1}\sigma^y_{4j+2}\sigma^x_{4j+3}$
$\sigma^x_{4j-2}1_{4j-1}1_{4j}\sigma^x_{4j+1}\sigma^z_{4j+2}\sigma^x_{4j+3}1_{4j+4}1_{4j+5}\sigma^x_{4j+6}$	
$i1_1\sigma^x_2$	$\varphi^{(3,1,3)}_{2j} = \frac{1}{\sqrt{2}}(\prod_{\nu=1}^{j} \sigma^x_{4\nu-3}\sigma^z_{4\nu-2}\sigma^x_{4\nu-1}1_{4\nu})1_{4j+1}\sigma^x_{4j+2}$
	$\varphi^{(3,1,3)}_{2j+1} = \frac{1}{\sqrt{2}}(\prod_{\nu=1}^{j} \sigma^x_{4\nu-3}\sigma^z_{4\nu-2}\sigma^x_{4\nu-1}1_{4\nu})\sigma^x_{4j+1}\sigma^y_{4j+2}\sigma^x_{4j+3}$

(4, 1, −)	
$\sigma^x_{4j-7}\sigma^z_{4j-6}\sigma^y_{4j-5}1_{4j-4}\sigma^y_{4j-3}\sigma^z_{4j-2}\sigma^x_{4j-1}$	
$\sigma^x_{4j-3}\sigma^z_{4j-2}\sigma^x_{4j-1}$	$\sigma^x_{4j-3}\sigma^z_{4j-2}\sigma^y_{4j-1}1_{4j}\sigma^z_{4j+1}$
$\sigma^z_{4j-1}1_{4j}\sigma^z_{4j+1}$	$\sigma^z_{4j-1}1_{4j}\sigma^y_{4j+1}\sigma^z_{4j+2}\sigma^x_{4j+3}$
$\sigma^z_{4j-1}1_{4j}\sigma^y_{4j+1}\sigma^z_{4j+2}\sigma^y_{4j+3}1_{4j+4}\sigma^z_{4j+5}$	
$-i\sigma^z_1$	$\varphi^{(4,1,-)}_{2j} = \frac{(-1)^{j-1}}{\sqrt{2}}(\prod_{\nu=1}^{j} \sigma^y_{4\nu-3}\sigma^z_{4\nu-2}\sigma^y_{4\nu-1}1_{4\nu})\sigma^z_{4j+1}$
	$\varphi^{(4,1,-)}_{2j+1} = \frac{(-1)^{j}}{\sqrt{2}}(\prod_{\nu=1}^{j} \sigma^y_{4\nu-3}\sigma^z_{4\nu-2}\sigma^y_{4\nu-1}1_{4\nu})\sigma^y_{4j+1}\sigma^z_{4j+2}\sigma^x_{4j+3}$

$(4,2,-)$

$\sigma^x_{6j-11}\sigma^z_{6j-10}\sigma^z_{6j-9}\sigma^y_{6j-8}1_{6j-7}1_{6j-6}\sigma^y_{6j-5}\sigma^z_{6j-4}\sigma^z_{6j-3}\sigma^x_{6j-2}$	
$\sigma^x_{6j-5}\sigma^z_{6j-4}\sigma^z_{6j-3}\sigma^x_{6j-2}$	$\sigma^x_{6j-5}\sigma^z_{6j-4}\sigma^z_{6j-3}\sigma^y_{6j-2}1_{6j-1}1_{6j}\sigma^z_{6j+1}$
$\sigma^z_{6j-2}\ 1_{6j-1}1_{6j}\sigma^z_{6j+1}$	$\sigma^z_{6j-2}1_{6j-1}1_{6j}\sigma^y_{6j+1}\sigma^z_{6j+2}\sigma^z_{6j+3}\sigma^x_{6j+4}$
$\sigma^z_{6j-2}1_{6j-1j}1_{6j}\sigma^y_{6j+1}\sigma^z_{6j+2}\sigma^z_{6j+3}\sigma^y_{6j+4}1_{6j+5}1_{6j+6}\sigma^z_{6j+7}$	

$-i\sigma^z_1$

$$\varphi^{(4,2,-)}_{2j} = \frac{(-1)^{j-1}}{\sqrt{2}}(\prod_{\nu=1}^{j}\sigma^y_{6\nu-5}\sigma^z_{6\nu-4}\sigma^z_{6\nu-3}\sigma^y_{6\nu-2}1_{6\nu-1}1_{6\nu})\sigma^z_{6j+1}$$

$$\varphi^{(4,2,-)}_{2j+1} = \frac{(-1)^{j}}{\sqrt{2}}(\prod_{\nu=1}^{j}\sigma^y_{6\nu-5}\sigma^z_{6\nu-4}\sigma^z_{6\nu-3}\sigma^y_{6\nu-2}1_{6\nu-1}1_{6\nu})\sigma^y_{6j+1}\sigma^z_{6j+2}\sigma^z_{6j+3}\sigma^x_{6j+4}$$

$(5,3,-)$

$\sigma^z_{3j-5}\sigma^z_{3j-4}\sigma^y_{3j-3}\sigma^y_{3j-2}\sigma^z_{3j-1}\sigma^z_{3j}$	
$\sigma^z_{3j-2}\sigma^z_{3j-1}\sigma^z_{3j}$	$\sigma^z_{3j-2}\sigma^z_{3j-1}\sigma^y_{3j}\sigma^x_{3j+1}$
$\sigma^x_{3j}\sigma^x_{3j+1}$	$\sigma^x_{3j}\sigma^y_{3j+1}\sigma^z_{3j+2}\sigma^z_{3j+3}$
$\sigma^x_{3j}\sigma^y_{3j+1}\sigma^z_{3j+2}\sigma^y_{3j+3}\sigma^x_{3j+4}$	

$i\sigma^x_1$

$$\varphi^{(5,3,-)}_{2j} = \frac{(-1)^{j}}{\sqrt{2}}(\prod_{\nu=1}^{j}\sigma^y_{3\nu-2}\sigma^z_{3\nu-1}\sigma^y_{3\nu})\sigma^x_{3j+1}$$

$$\varphi^{(5,3,-)}_{2j+1} = \frac{(-1)^{j}}{\sqrt{2}}(\prod_{\nu=1}^{j}\sigma^y_{3\nu-2}\sigma^z_{3\nu-1}\sigma^y_{3\nu})\sigma^y_{3j+1}\sigma^z_{3j+2}\sigma^z_{3j+3}$$

$(1,2,1)$

$\sigma^x_{j-1}1_j\sigma^z_{j+1}\sigma^z_{j+2}1_{j+3}\sigma^x_{j+4}$	
$\sigma^x_j\sigma^x_{j+1}\sigma^z_{j+2}\sigma^x_{j+3}\sigma^x_{j+4}$	$\sigma^x_j\sigma^x_{j+1}\sigma^y_{j+2}1_{j+3}\sigma^x_{j+4}$
$\sigma^x_{j+2}\sigma^x_{j+3}$	$\sigma^x_{j+2}1_{j+3}\sigma^y_{j+4}\sigma^x_{j+5}\sigma^x_{j+6}$
$\sigma^x_{j+2}1_{j+3}\sigma^z_{j+4}1_{j+5}\sigma^x_{j+6}$	

$i\sigma^y_1\sigma^z_2\sigma^x_3\sigma^x_4$

$$\varphi^{(1,2,1)}_{2j} = \frac{1}{\sqrt{2}}(\prod_{\nu=1}^{j}\sigma^z_\nu)\sigma^y_{j+1}\sigma^z_{j+2}\sigma^x_{j+3}\sigma^x_{j+4}$$

$$\varphi^{(1,2,1)}_{2j+1} = \frac{1}{\sqrt{2}}(\prod_{\nu=1}^{j}\sigma^z_\nu)\sigma^z_{j+1}\sigma^y_{j+2}\sigma^y_{j+3}1_{j+4}\sigma^x_{j+5}$$

$(2,2,1)$

$\sigma^x_{5j-9}\sigma^x_{5j-8}\sigma^z_{5j-7}\sigma^x_{5j-6}\sigma^z_{5j-5}\sigma^z_{5j-4}\sigma^x_{5j-3}\sigma^z_{5j-2}\sigma^x_{5j-1}\sigma^x_{5j}$	
$\sigma^x_{5j-4}\sigma^x_{5j-3}\sigma^z_{5j-2}\sigma^x_{5j-1}\sigma^x_{5j}$	$\sigma^x_{5j-4}\sigma^x_{5j-3}\sigma^z_{5j-2}\sigma^x_{5j-1}\sigma^z_{5j}\sigma^y_{5j+1}$
$\sigma^y_{5j}\sigma^y_{5j+1}$	$\sigma^y_{5j}\sigma^z_{5j+1}\sigma^x_{5j+2}\sigma^z_{5j+3}\sigma^x_{5j+4}\sigma^x_{5j+5}$
$\sigma^y_{5j}\sigma^z_{5j+1}\sigma^x_{5j+2}\sigma^z_{5j+3}\sigma^x_{5j+4}\sigma^z_{5j+5}\sigma^y_{5j+6}$	

$i\sigma^y_1$

$$\varphi^{(2,2,1)}_{2j} = \frac{(-1)^{j}}{\sqrt{2}}(\prod_{\nu=1}^{j}\sigma^z_{5\nu-4}\sigma^x_{5\nu-3}\sigma^z_{5\nu-2}\sigma^x_{5\nu-1}\sigma^z_{5\nu})\sigma^y_{5j+1}$$

$$\varphi^{(2,2,1)}_{2j+1} = \frac{(-1)^{j}}{\sqrt{2}}(\prod_{\nu=1}^{j}\sigma^z_{5\nu-4}\sigma^x_{5\nu-3}\sigma^z_{5\nu-2}\sigma^x_{5\nu-1}\sigma^z_{5\nu})\sigma^z_{5j+1}\sigma^x_{5j+2}\sigma^z_{5j+3}\sigma^x_{5j+4}\sigma^x_{5j+5}$$

$(4,2,2)$	
$\sigma^x_{3j-5}\sigma^x_{3j-4}\sigma^y_{3j-3}1_{3j-2}1_{3j-1}\sigma^y_{3j}\sigma^x_{3j+1}\sigma^x_{3j+2}$	
$\sigma^x_{3j-2}\sigma^x_{3j-1}\sigma^z_{3j}\sigma^x_{3j+1}\sigma^x_{3j+2}$	$\sigma^x_{3j-2}\sigma^x_{3j-1}\sigma^y_{3j}\sigma^x_{3j+1}\sigma^x_{3j+2}\sigma^x_{3j+3}$
$\sigma^x_{3j}1_{3j+1}1_{3j+2}\sigma^x_{3j+3}$	$\sigma^x_{3j}\sigma^x_{3j+1}\sigma^x_{3j+2}\sigma^y_{3j+3}\sigma^x_{3j+4}\sigma^x_{3j+5}$
$\sigma^x_{3j}\sigma^x_{3j+1}\sigma^x_{3j+2}\sigma^z_{3j+3}\sigma^x_{3j+4}\sigma^x_{3j+5}\sigma^x_{3j+6}$	
$i\sigma^x_1\sigma^x_2\sigma^x_3$	$\varphi^{(4,2,2)}_{2j} = \frac{1}{\sqrt{2}}(\prod_{\nu=1}^{j} 1_{3\nu-2}1_{3\nu-1}\sigma^z_{3\nu})\sigma^x_{3j+1}\sigma^x_{3j+2}\sigma^x_{3j+3}$
	$\varphi^{(4,2,2)}_{2j+1} = \frac{1}{\sqrt{2}}(\prod_{\nu=1}^{j} 1_{3\nu-2}1_{3\nu-1}\sigma^z_{3\nu})1_{3j+1}1_{3j+2}\sigma^y_{3j+3}\sigma^x_{3j+4}\sigma^x_{3j+5}$

$(11,-,-)$	
$\sigma^x_{4j-7}\sigma^x_{4j-6}\sigma^z_{4j-5}1_{4j-4}\sigma^z_{4j-3}1_{4j-2}\sigma^z_{4j-1}\sigma^x_{4j}\sigma^x_{4j+1}$	
$\sigma^x_{4j-3}\sigma^x_{4j-2}\sigma^z_{4j-1}\sigma^x_{4j}\sigma^x_{4j+1}$	$\sigma^x_{4j-3}\sigma^x_{4j-2}\sigma^z_{4j-1}1_{4j}\sigma^y_{4j+1}\sigma^x_{4j+2}$
$\sigma^x_{4j}\sigma^z_{4j+1}\sigma^x_{4j+2}$	$\sigma^x_{4j}\sigma^y_{4j+1}1_{4j+2}\sigma^z_{4j+3}\sigma^x_{4j+4}\sigma^x_{4j+5}$
$\sigma^x_{4j}\sigma^y_{4j+1}1_{4j+2}\sigma^z_{4j+3}1_{4j+4}\sigma^y_{4j+5}\sigma^x_{4j+6}$	
$i\sigma^y_1\sigma^x_2$	$\varphi^{(11,-,-)}_{2j} = \frac{1}{\sqrt{2}}(\prod_{\nu=1}^{j} \sigma^z_{4\nu-3}1_{4\nu-2}\sigma^z_{4\nu-1}1_{4\nu})\sigma^y_{4j+1}\sigma^x_{4j+2}$
	$\varphi^{(11,-,-)}_{2j} = \frac{1}{\sqrt{2}}(\prod_{\nu=1}^{j} \sigma^z_{4\nu-3}1_{4\nu-2}\sigma^z_{4\nu-1}1_{4\nu})\sigma^z_{4j+1}1_{4j+2}\sigma^z_{4j+3}\sigma^x_{4j+4}\sigma^x_{4j+5}$

Minami, Nucl. Phys.B 2017

Hamiltonian (generalized XY-chain)

$$-\beta\mathcal{H} = K_{-m}\sum_{j=1}^{N}\varphi_2(j)\varphi_1(j+\kappa-m) + K_0\sum_{j=1}^{N}\varphi_2(j)\varphi_1(j+\kappa) + K_m\sum_{j=1}^{N}\varphi_2(j)\varphi_1(j+\kappa+m).$$

Ground state correlation function

$$\langle(-2i)\varphi_2(j)\varphi_1(j+\kappa)\rangle_0 = \frac{1}{2\pi}\int_0^{\pi}\frac{Le^{iq\kappa}+L^{\dagger}e^{-iq\kappa}}{\sqrt{LL^{\dagger}}}\,dq, \qquad L=\sum_l K_l e^{iql}$$

String correlation function

$$I_n = \langle\prod_{j=j_0}^{j_0+n-1}(-2i)\varphi_2(j)\varphi_1(j+\kappa)\rangle_0 = (-2i)^n \det M_n,$$

Wick theorem

$n\to\infty$ limit and exponent

Szego theorem

$$\lim_{n\to\infty}\frac{\det M_n}{\mu^n} = \exp\Big(\sum_{n=1}^{\infty} n g_n g_{-n}\Big) = \begin{cases} \pm[(1-a_m^2)(1-a_{-m}^2)/(1-a_m a_{-m})]^{|m|/4} & |a_m|<1 \\ 0 & |a_m|>1 \end{cases}$$

Central charge $\quad c = m/2$

exponent |m|/4

Ex. transverse Ising model

$$\langle\sigma_j^z\rangle_0^2 = \lim_{n\to\infty}\langle\sigma_j^z\sigma_{j+n}^z\rangle_0 \simeq \pm[(1-K_m/K_0)(1+K_m/K_0)]^{1/4} \qquad \beta = 1/8.$$

That of 2-dim Ising model

Central charge

Estimation of the conformal charge from finite-size correction
Dispersion

$$\Lambda(q) = 2\sqrt{LL^{\dagger}} \simeq 2|K|m\gamma\,|q| \quad (\frac{l}{N}\pi = q \simeq 0) \tag{1}$$

Then conformal invariant normalization is

$$2|K|m\gamma = 1 \tag{2}$$

Consider finite size correction of the ground state energy

$$E_0 \quad = \quad -\sum_{0<q<\pi} 2|K|2\gamma|\sin\frac{ml\pi}{2N}|[1-\frac{\gamma^2-1}{\gamma^2}\sin^2\frac{ml\pi}{2N}]^{1/2} \tag{3}$$

The term proportional to $1/N$ is obtained as

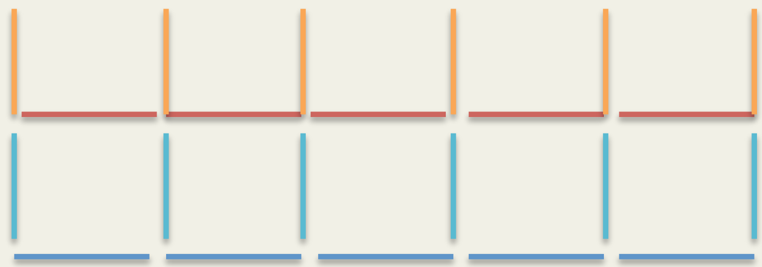
$$-2|K|\gamma m\,\frac{1}{6}\frac{m\pi}{2N} = -\frac{m\pi}{12N} \quad (= -\frac{c\pi}{6N}) \tag{4}$$

Conformal charge is

$$c = m/2 \qquad (m = 1, 2, 3, \ldots) \tag{5}$$

Two-dimensional classical systems

2-dim. Ising model（period m）



$$-\beta\mathcal{H} = \sum_{i=1}^{M}\sum_{j=1}^{N}(K_{1i}\sigma^z_{ij}\sigma^z_{i+1j} + K_{2i}\sigma^z_{ij}\sigma^z_{ij+1})$$

Partition function

$$Z = \mathrm{tr}\Big(\prod_{i=1}^{m} V_{1i}V_{2i}\Big)^{M/m}$$

Transfer matrix

$$V_{1i} = \Big(\frac{e^{K_{1i}}}{\cosh K^*_{1i}}\Big)^N \exp(K^*_{1i}\sum_{j=1}^{N}\sigma^x_j) \qquad V_{2i} = \exp(K_{2i}\sum_{j=1}^{N}\sigma^z_j\sigma^z_{j+1})$$

$$= \Big(\frac{e^{K_{1i}}}{\cosh K^*_{1i}}\Big)^N \prod_{0<q<\pi}\exp(K^*_{1i}(-2i)C_{12}(q)) \qquad = \prod_{0<q<\pi}\exp(K_{2i}(-2i)\tilde{C}_{21}(q))$$

where

$$\begin{aligned} C_{12}(q) &= C_1^\dagger(q)C_2(q) + C_1(q)C_2^\dagger(q) \\ \tilde{C}_{21}(q) &= e^{iq}C_2^\dagger(q)C_1(q) + e^{-iq}C_2(q)C_1^\dagger(q) \end{aligned} \qquad C^3 = -C$$

Transformation

$$\varphi_1(j) = \frac{1}{\sqrt{2}}(\prod_{k=1}^{j}\sigma^x_k)\,\sigma^z_{j+1} \qquad \varphi_2(j) = \frac{-1}{\sqrt{2}}(\prod_{k=1}^{j}\sigma^x_k)\,\sigma^y_{j+1}$$

Transfer matrix $V = \prod_{i=1}^{m} V_{1i} V_{2i}$
(represented by $|0\,0\rangle$**,** $|1\,0\rangle = C_1^\dagger(q)|0\,0\rangle$**,** $|0\,1\rangle = C_2^\dagger(q)|0\,0\rangle$ $|1\,1\rangle = C_2^\dagger(q)C_1^\dagger(q)|0\,0\rangle$**)**

$$V = \prod_{i=1}^{m} \Big(\frac{e^{K_{1i}}}{\cosh K_{1i}^*}\Big)^N \prod_{0<q<\pi} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A_i(q) & \bar{B}_i(q) & 0 \\ 0 & B_i(q) & \bar{A}_i(q) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where

$$\begin{aligned} A_i(q) &= \cosh 2K_{1i}^* \cosh 2K_{2i} - e^{iq} \sinh 2K_{1i}^* \sinh 2K_{2i} \\ B_i(q) &= i(\sinh 2K_{1i}^* \cosh 2K_{2i} - e^{iq} \cosh 2K_{1i}^* \sinh 2K_{2i}) \end{aligned}$$

maximum eigenvalue is obtained from the block-element

$$v(q) = \prod_{i=1}^{m} \begin{pmatrix} A_i(q) & \bar{B}_i(q) \\ B_i(q) & \bar{A}_i(q) \end{pmatrix}, \qquad 0 = \lambda^2 - (\mathrm{tr}\, v(q))\lambda + \det v(q).$$

$$\cosh m\epsilon_q = \frac{1}{2} \mathrm{tr}\, v(q) \qquad \lambda = e^{m\epsilon_q}, e^{-m\epsilon_q} \qquad \epsilon_q > 0$$

The free energy is

$$
\begin{aligned}
-\beta f &= \lim_{(N,M)\to(\infty,\infty)} \frac{1}{NM} \log Z \\
&= \lim_{N\to\infty} \frac{1}{N} \Big[\frac{1}{m} \log \prod_{i=1}^{m} \Big(\frac{e^{K_{1i}}}{\cosh K_{1i}^*} \Big)^N + \sum_{0<q<\pi} \epsilon_q \Big] \\
&= \frac{1}{m} \log \prod_{i=1}^{m} \Big(\frac{e^{K_{1i}}}{\cosh K_{1i}^*} \Big) + \frac{1}{2\pi} \int_0^\pi \epsilon_q \, dq.
\end{aligned}
$$

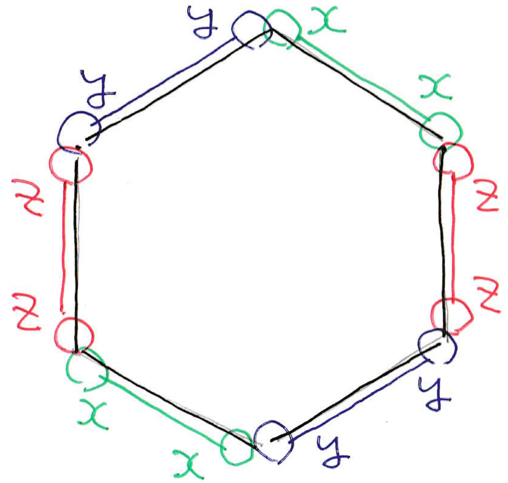
When m=1, $K_{1i} = K_1 \;\; K_{2i} = K_2$**, then**

$$
\begin{aligned}
-\beta f &= \log \Big(\frac{e^{K_1}}{\cosh K_1^*} \Big) + \frac{1}{2\pi} \int_0^\pi \epsilon_q \, dq, \\
\cosh \epsilon_q &= \cosh 2K_1^* \cosh 2K_2 - \sinh 2K_1^* \sinh 2K_2 \cos q.
\end{aligned}
$$

(Onsager 1944)

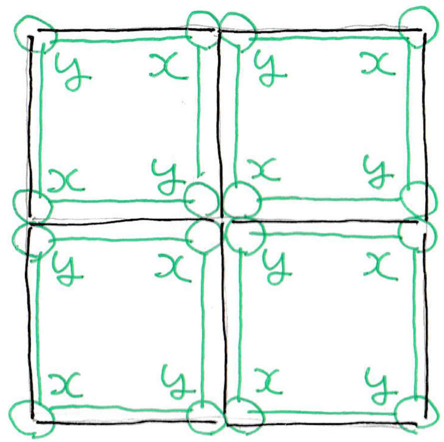
two-dimensional quantum systems

$$x = \sigma_j^x$$
$$y = \sigma_j^y$$
$$z = \sigma_j^z$$



Kitaev model

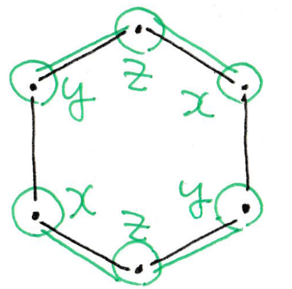
$$-\beta\mathcal{H} = \sum [K_1 \sigma_i^x \sigma_j^x + K_2 \sigma_k^y \sigma_l^y + K_3 \sigma_m^z \sigma_n^z]$$



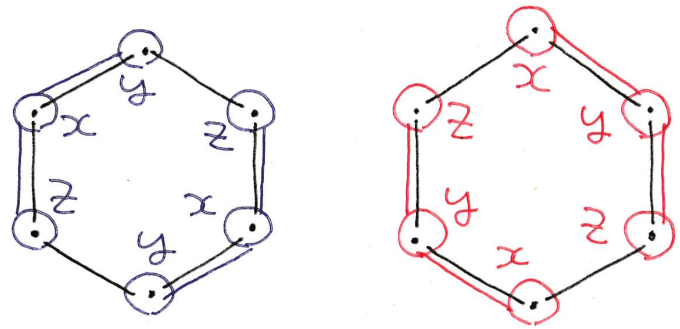
Wen model

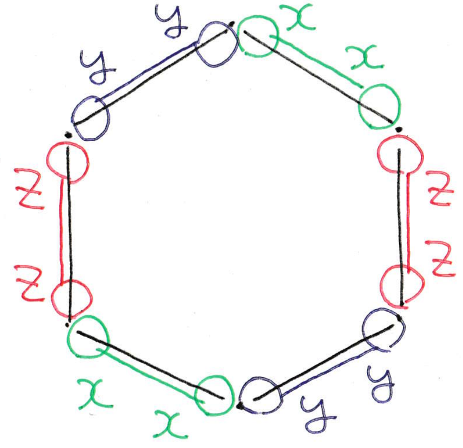
$$-\beta\mathcal{H} = K \sum_{ij} \sigma_{ij}^x \sigma_{ij+1}^y \sigma_{i+1j}^x \sigma_{i+1j+1}^y$$

Lee et el. 2007
Si et al. 2009

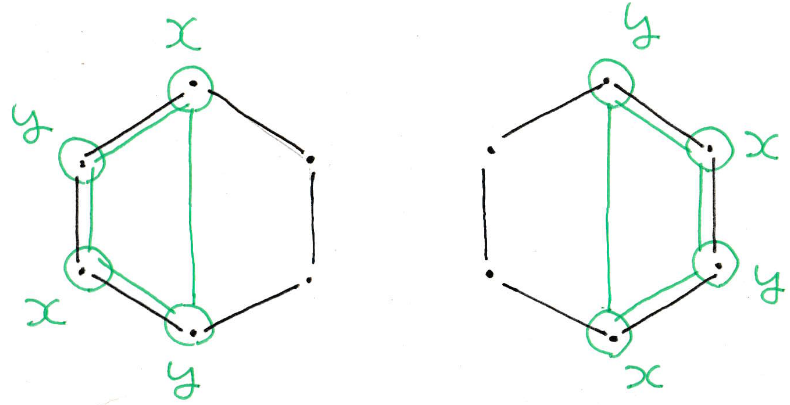


Yu 2008
Yu Wang 2008

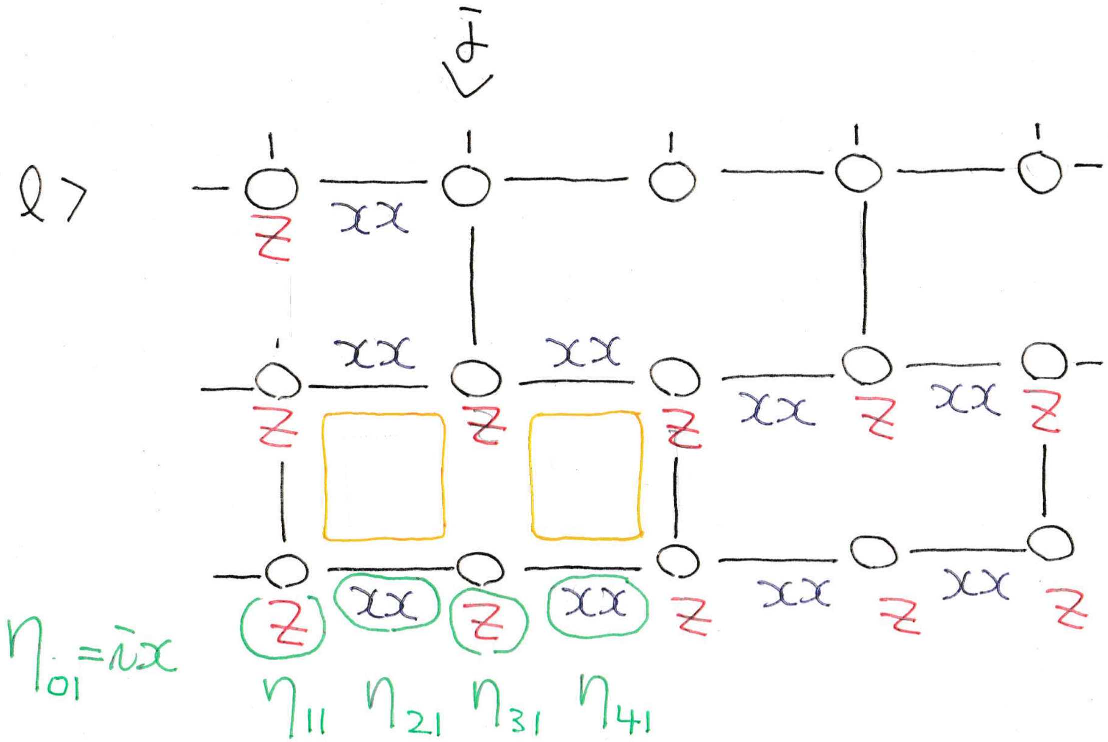




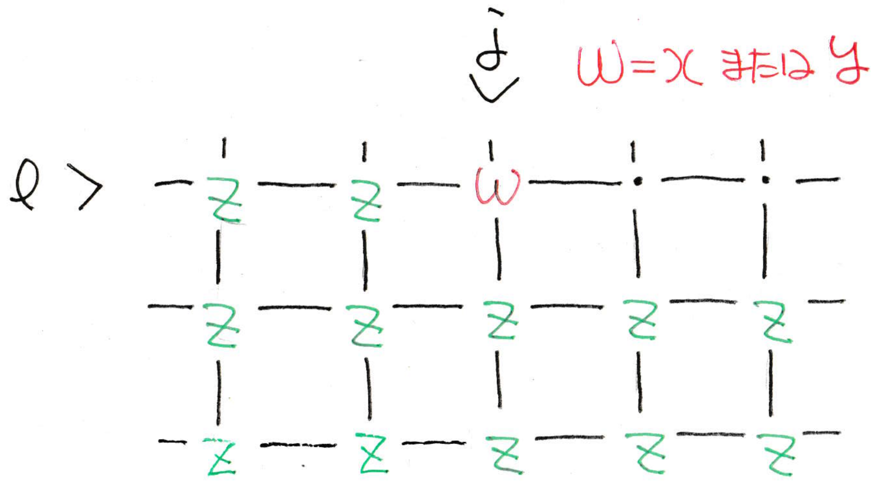
Kitaev 2006

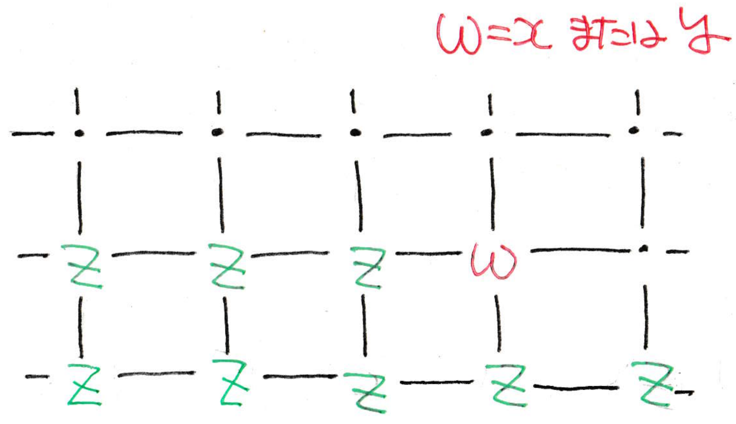


4-body terms (Wen model)



Series of operators in two-dimension





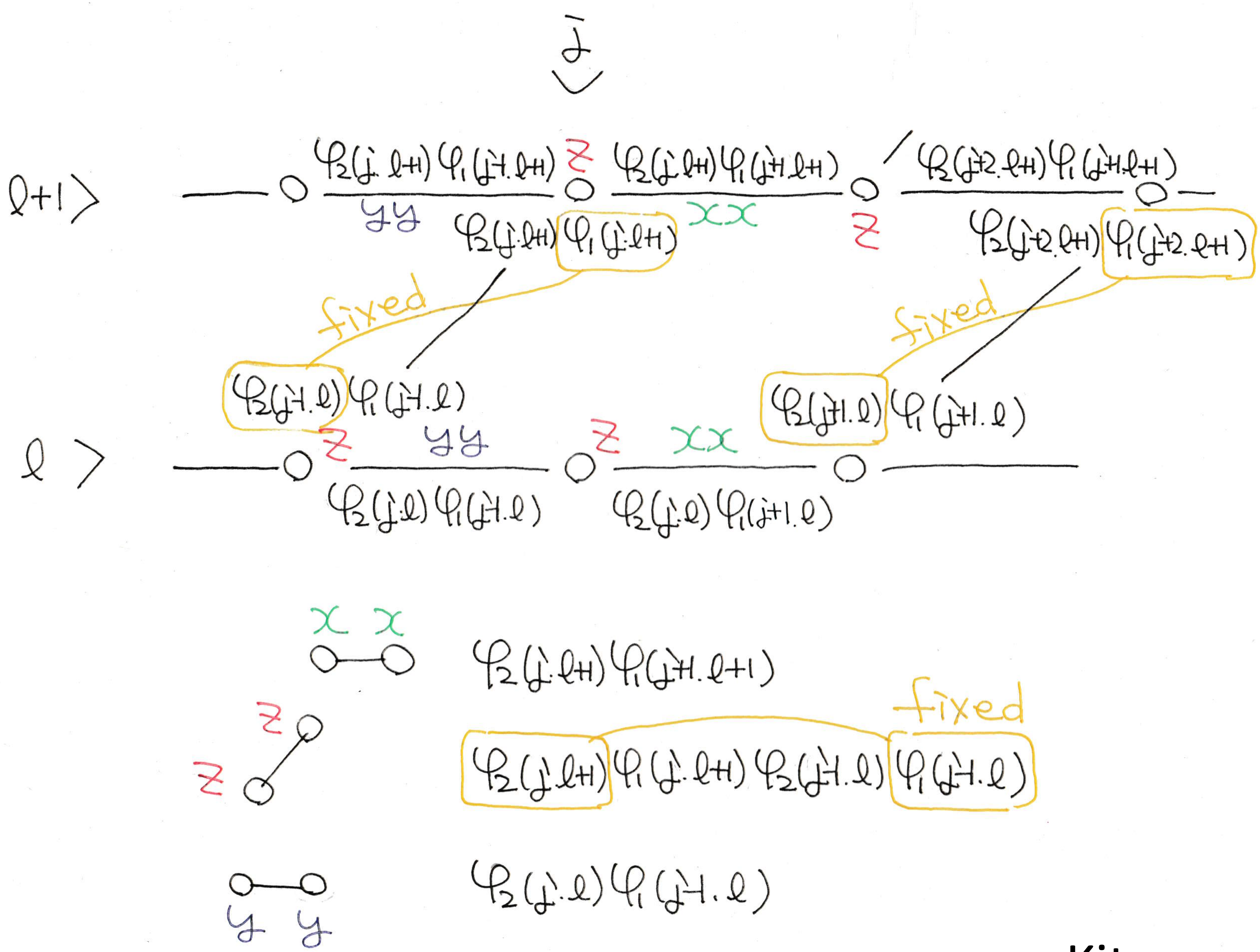
$\varphi_K(j,l) =$

$\varphi_{K'}(j',l') =$

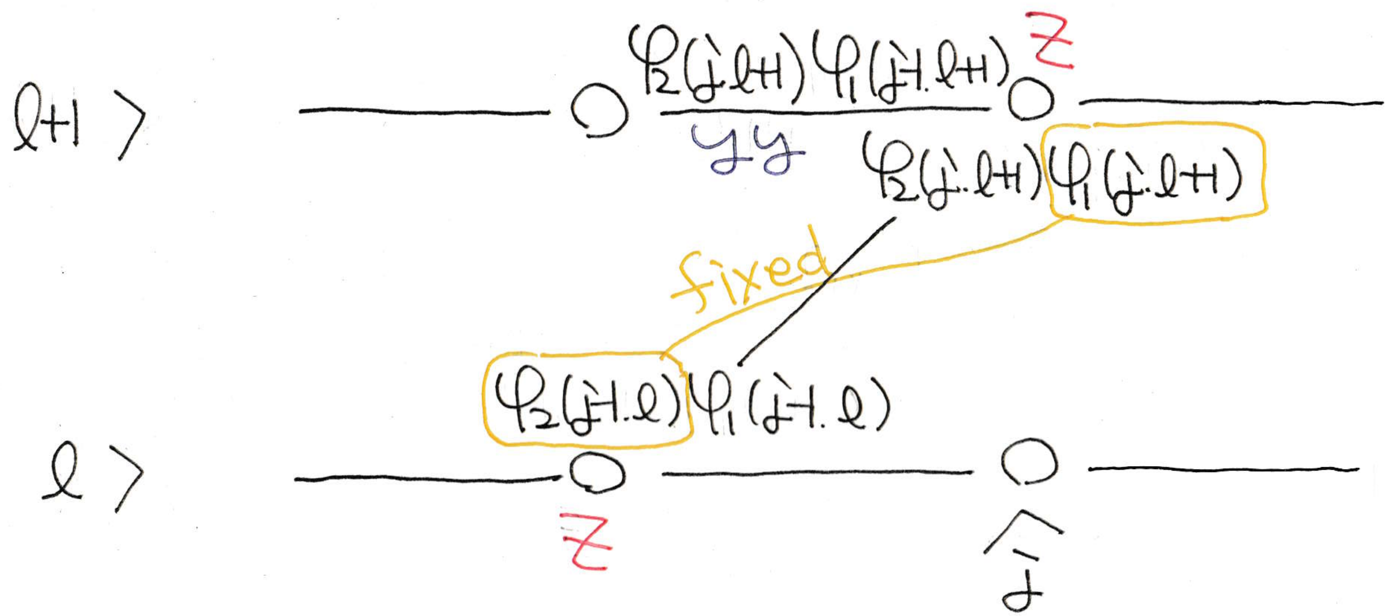
$$\{\varphi_K(j,l), \varphi_{K'}(j',l')\} = \delta_{jj'}\delta_{ll'}$$

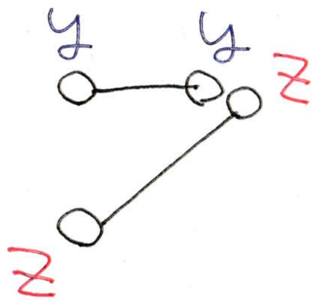
Two-dimesnional
Jordan-Wigner transformation

Feng et al. 2007



Kitaev model





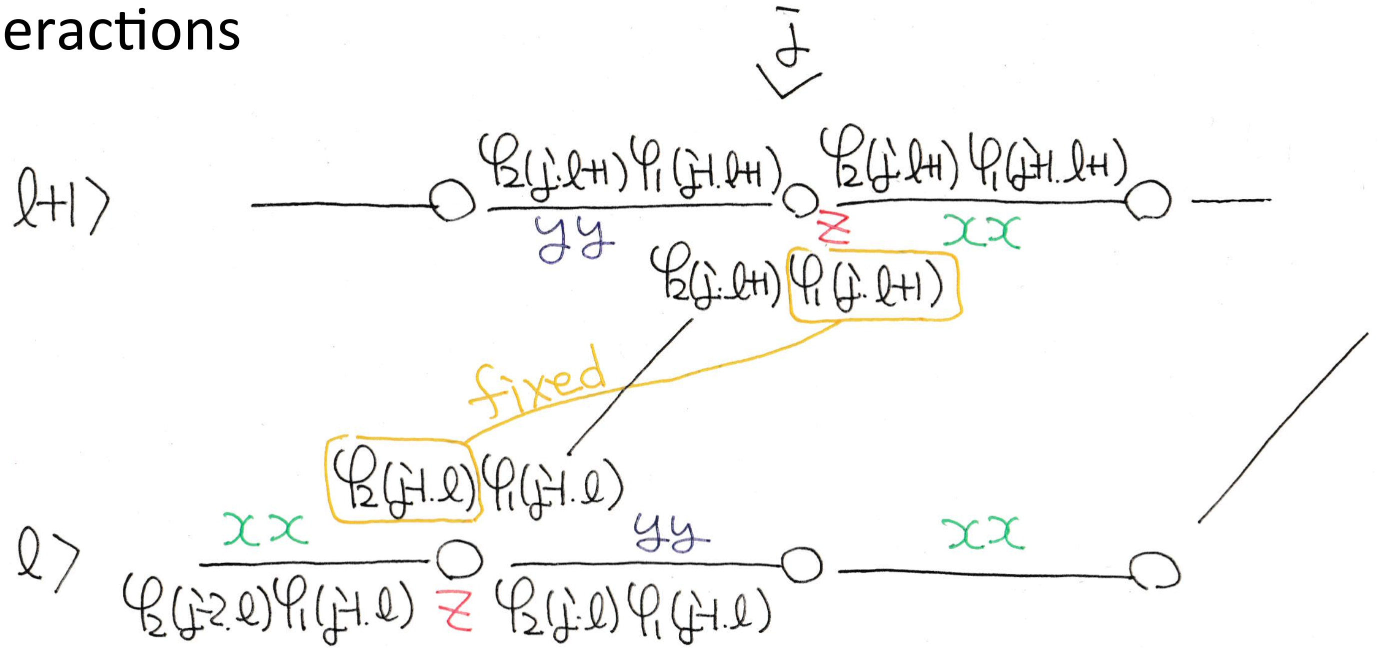
fixed

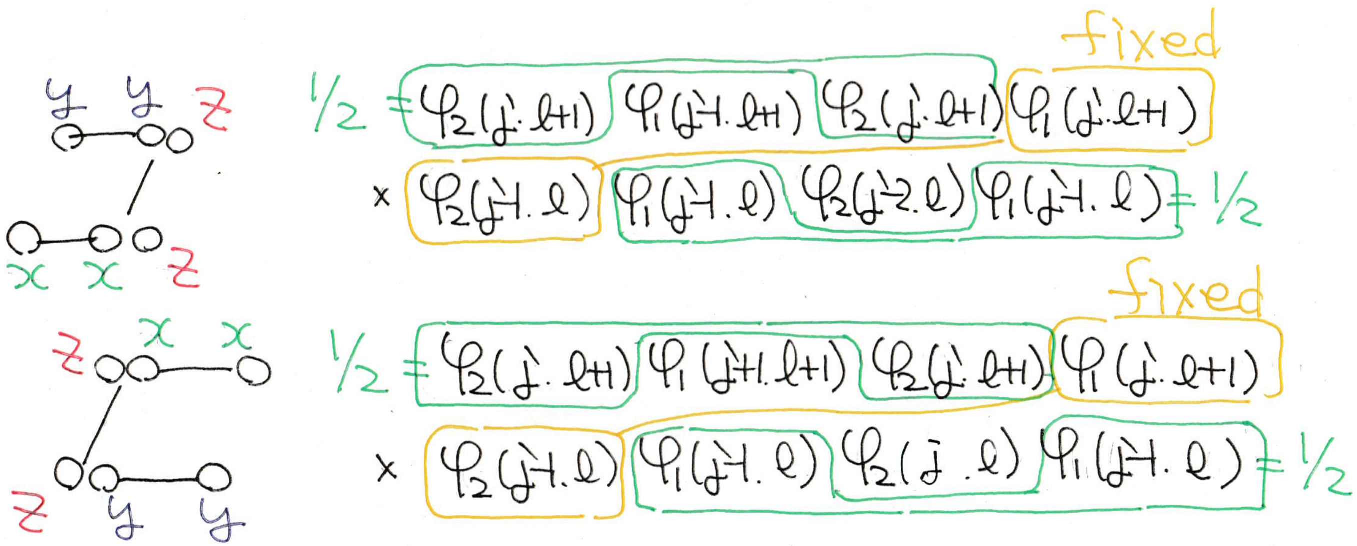
$$\frac{1}{2} = \varphi_2(j, l+1)\,\varphi_1(j-1, l+1)\,\varphi_2(j, l+1)\,\varphi_1(j, l+1)$$

$$\times\ \varphi_2(j-1, l)\,\varphi_1(j-1, l)$$

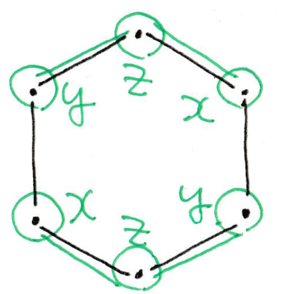
Six kind of three-body

Wen interactions

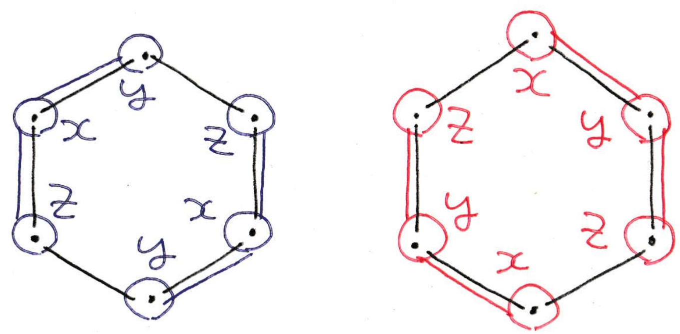


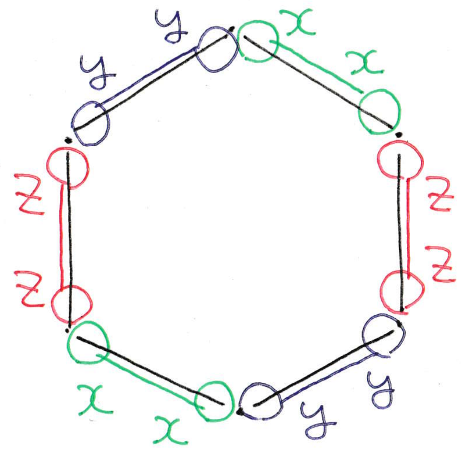


Lee et el. 2007
Si et al. 2009

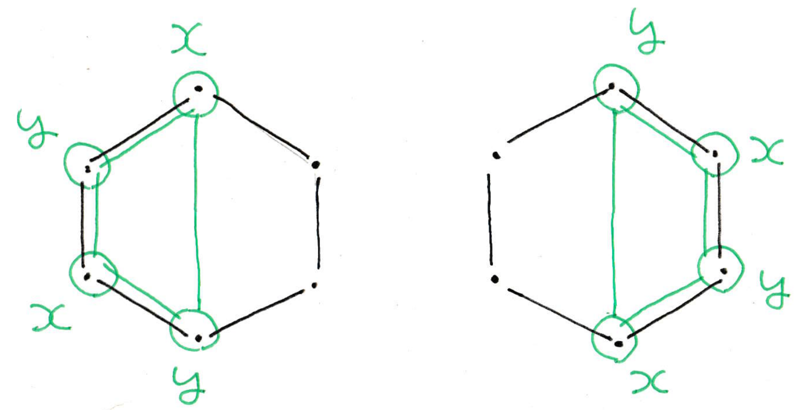


Yu 2008
Yu Wang 2008

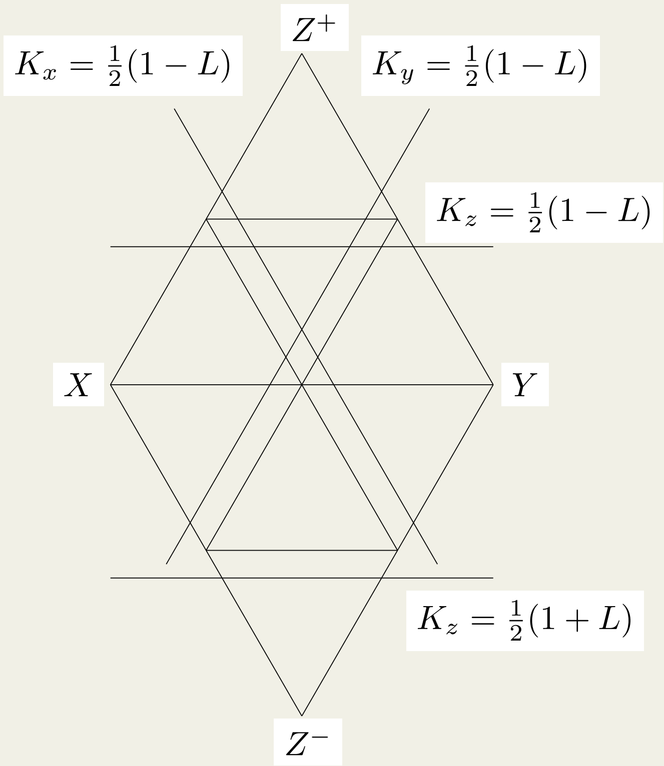


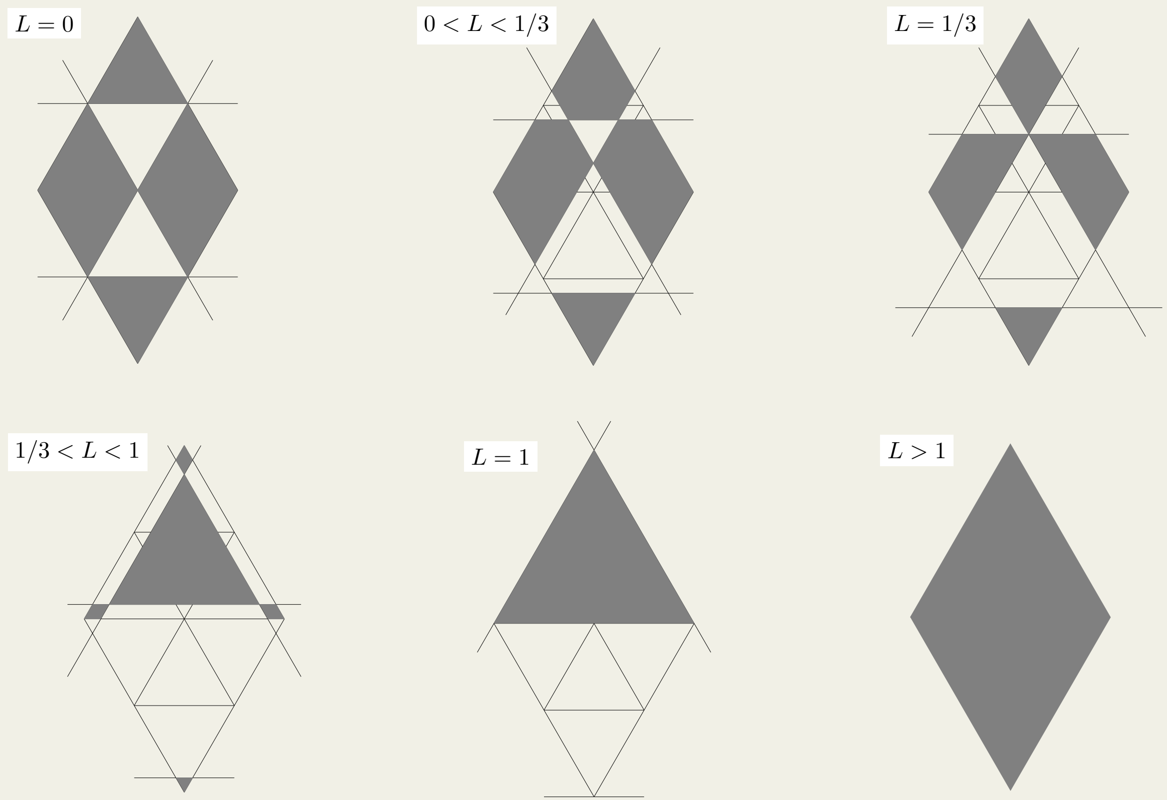


Kitaev 2006



4-body terms (Wen model)





		(K_x, K_y, K_z)
X	:	$(1, 0, 0)$
Y	:	$(0, 1, 0)$
Z^+	:	$(0, 0, 1)$
Z^-	:	$(0, 0, -1)$

Symmetry of the system

$$
\begin{aligned}
&(a)\ \ (K_x,K_y)\mapsto(-K_x,-K_y)\quad (\sigma^x_{jl},\sigma^y_{jl},\sigma^z_{jl})\mapsto(-\sigma^x_{jl},-\sigma^y_{jl},\sigma^z_{jl})\ \ \text{if}\ \ j=\text{odd},\\
&(b)\ \ (L,K_z)\mapsto(-L,-K_z)\quad (\sigma^x_{jl},\sigma^y_{jl},\sigma^z_{jl})\mapsto(\sigma^x_{jl},-\sigma^y_{jl},-\sigma^z_{jl})\ \ \text{if}\ \ l=\text{odd},\\
&(c)\ \ (K_x,K_z)\mapsto(-K_x,-K_z)\quad (\sigma^x_{jl},\sigma^y_{jl},\sigma^z_{jl})\mapsto(-\sigma^x_{jl},\sigma^y_{jl},-\sigma^z_{jl})\ \ \text{if}\ \ j=\text{odd},\\
&(d)\ \ (K_y,K_z)\mapsto(-K_y,-K_z)\quad (\sigma^x_{jl},\sigma^y_{jl},\sigma^z_{jl})\mapsto(\sigma^x_{jl},-\sigma^y_{jl},-\sigma^z_{jl})\ \ \text{if}\ \ j=\text{odd},\\
&(e)\ \ (L,K_x)\mapsto(-L,-K_x)\quad (\sigma^x_{jl},\sigma^y_{jl},\sigma^z_{jl})\mapsto(-\sigma^x_{jl},\sigma^y_{jl},-\sigma^z_{jl})\\
&\qquad\qquad\qquad\qquad\qquad\text{if}\ \ (j,l)=(\text{odd},\text{even})\ \ \text{or}\ \ (\text{even},\text{odd}),\\
&(f)\ \ (L,K_y)\mapsto(-L,-K_y)\quad (\sigma^x_{jl},\sigma^y_{jl},\sigma^z_{jl})\mapsto(\sigma^x_{jl},-\sigma^y_{jl},-\sigma^z_{jl})\\
&\qquad\qquad\qquad\qquad\qquad\text{if}\ \ (j,l)=(\text{odd},\text{even})\ \ \text{or}\ \ (\text{even},\text{odd}),
\end{aligned}
$$

$\mathcal{H}(\cdots,K_i,K_j,\cdots)$ **and** $\mathcal{H}(\cdots,-K_i,-K_j,\cdots)$ **are equivalent.**

$$
\begin{aligned}
&(q_1,q_2)\mapsto(-q_1,-q_2),\qquad \text{and}\\
&(K_x,K_y)\mapsto(K_y,K_x),\quad (K_z,L)\mapsto(L,K_z),\\
&\tilde{c}^\dagger_1(q_1,q_2)=c^\dagger_1(-q_1,-q_2)\quad(=c_1(q_1,q_2)),\\
&\tilde{c}_2(q_1,q_2)=c_2(-q_1,-q_2)\quad(=c^\dagger_2(q_1,q_2)).
\end{aligned}
$$

$\mathcal{H}(K_x,K_y,K_z,L)$ **and** $\mathcal{H}(K_y,K_x,L,K_z)$ **are identical as an operator.**

$$
\begin{aligned}
&(q_1,q_2)\mapsto(q_1,q)\qquad \text{where}\ \ q=q_1+q_2,\\
&(K_x,K_y)\mapsto(K_z,L),\quad (K_z,L)\mapsto(K_x,K_y),\\
&c^\dagger_1(q_1,q-q_1)=\tilde{c}^\dagger_1(q_1,q),\\
&c_2(q_1,q-q_1)=\tilde{c}_2(q_1,q).
\end{aligned}
$$

$\mathcal{H}(K_x,K_y,K_z,L)$ **and** $\mathcal{H}(K_z,L,K_x,K_y)$ **are identical as an operator.**

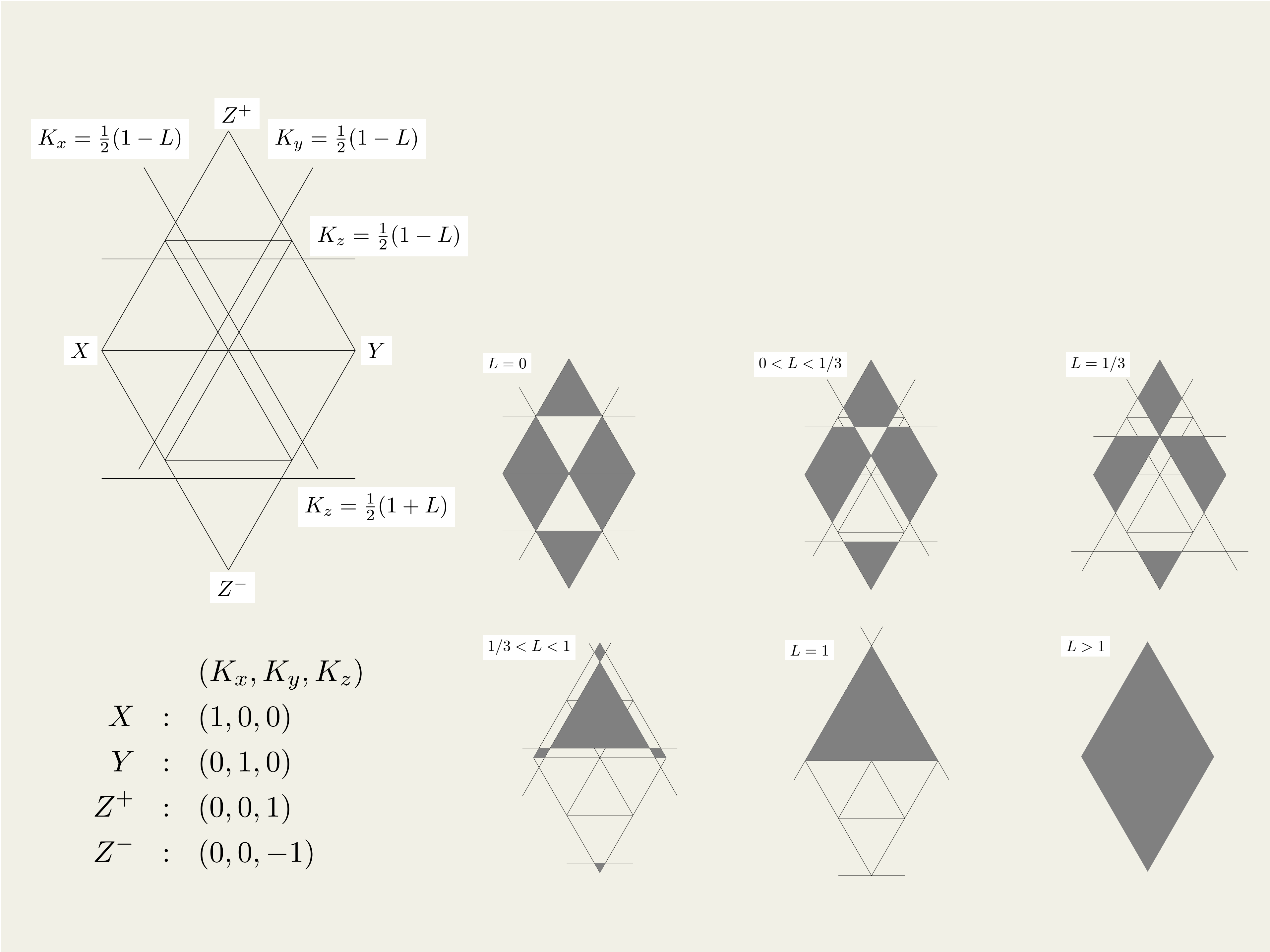
Anyon generator changes its location

The spin operators can be expressed by $\varphi_\alpha(j,l)$ as

$$\begin{aligned}
\sigma^z_{jl} &= \eta_{2j-1\,l} = (+2i)\varphi_2(j,l)\varphi_1(j,l),\\
\sigma^x_{jl} &= \sqrt{2}\Big(\prod_{r=1}^{l-1}\prod_{k=1}^{N}\eta_{2k-1\,r}\Big)\Big(\prod_{k=1}^{j-1}\eta_{2k-1\,l}\Big)\varphi_1(j,l),\\
\sigma^y_{jl} &= \sqrt{2}\Big(\prod_{r=1}^{l-1}\prod_{k=1}^{N}\eta_{2k-1\,r}\Big)\Big(\prod_{k=1}^{j-1}\eta_{2k-1\,l}\Big)\varphi_2(j,l),
\end{aligned}$$

where $\eta_{2k-1\,r}$ are also written by $\varphi_\alpha(j,l)$. The external field and string operators are, therefore, transformed together with $\varphi_\alpha(j,l)$.

We also find that the anyon excitations appear in all of the regions shown in the phase diagram, and they can be transformed each other.



Z^+
$K_x = \frac{1}{2}(1-L)$
$K_y = \frac{1}{2}(1-L)$
$K_z = \frac{1}{2}(1-L)$
X
Y
$K_z = \frac{1}{2}(1+L)$
Z^-
$L=0$
$0<L<1/3$
$L=1/3$
$1/3<L<1$
$L=1$
$L>1$
(K_x, K_y, K_z)
X : $(1,0,0)$
Y : $(0,1,0)$
Z^+ : $(0,0,1)$
Z^- : $(0,0,-1)$

Algebraic structure

Onsager Algebra

In his solution of the 2-dimensional Ising model, Onsager introduced

$$A_n = \sum_{j=1}^{N} \sigma_j^x \Big(\prod_{k=j+1}^{j+n-1} \sigma_k^z \Big) \sigma_{j+n}^x,$$

$$G_n = \frac{1}{2}\mathrm{i} \sum_{j=1}^{N} \Big[\sigma_j^x \Big(\prod_{k=j+1}^{j+n-1} \sigma_k^z \Big) \sigma_{j+n}^y + \sigma_j^y \Big(\prod_{k=j+1}^{j+n-1} \sigma_k^z \Big) \sigma_{j+n}^x \Big].$$

We assume periodicity $\sigma_{j\pm N}^\alpha = \sigma_j^\alpha$**,** $\alpha = x, y, z$**, and** $(\sigma_k^z)^2 = 1$**, we have**

$$\prod_{k=j+1}^{j} \sigma_k^z = P, \qquad \prod_{k=j+1}^{j-m} \sigma_k^z = P \prod_{k=j-m+1}^{j} \sigma_k^z,$$

so that, using this and the Pauli matrix product rules, we find

$$A_0 = -\sum_{j=1}^{N} \sigma_j^z, \quad A_{-n} = \sum_{j=1}^{N} \sigma_j^y \Big(\prod_{k=j+1}^{j+n-1} \sigma_k^z \Big) \sigma_{j+n}^y, \quad P \equiv \prod_{k=1}^{N} \sigma_k^z,$$

$$A_{n\pm N} = -PA_n = -A_nP, \quad A_{n\pm 2N} = A_n,$$

$$G_0 = 0, \quad G_{-n} = -G_n, \quad G_{n\pm N} = -PG_n = -G_nP, \quad G_{n\pm 2N} = G_n.$$

Onsager derived the following commutation rules:

$$[A_j, A_k] = 4G_{j-k}, \quad [G_m, A_l] = 2A_{l+m} - 2A_{l-m}, \quad [G_j, G_k] = 0.$$

From these we also have

$$[A_j, [A_j, [A_j, A_k]]] = 16[A_j, A_k], \quad [A_j, [A_j, G_k]] = 16G_k,$$

演算子列からの構成

$$A_1 = \sum_{j=1}^{N} \eta_{2j} \qquad A_2 = \sum_{j=1}^{N} \eta_{2j}\eta_{2j+1}\eta_{2j+2} \qquad A_3 = \sum_{j=1}^{N} \eta_{2j}\eta_{2j+1}\eta_{2j+2}\eta_{2j+3}\eta_{2j+4}$$

$$A_0 = \sum_{j=1}^{N} \eta_{2j-1} \quad A_{-1} = \sum_{j=1}^{N} \eta_{2j-3}\eta_{2j-2}\eta_{2j-1} \quad A_{-2} = \sum_{j=1}^{N} \eta_{2j-5}\eta_{2j-4}\eta_{2j-3}\eta_{2j-2}\eta_{2j-1}$$

$$G_0 = 0 \qquad G_1 = \frac{1}{2}\sum_{j=1}^{N}(\eta_{2j}\eta_{2j+1} - \eta_{2j-1}\eta_{2j})$$

$$G_2 = \frac{1}{2}\sum_{j=1}^{N}(\eta_{2j}\eta_{2j+1}\eta_{2j+2}\eta_{2j+3} - \eta_{2j-1}\eta_{2j}\eta_{2j+1}\eta_{2j+2})$$

$$G_3 = \frac{1}{2}\sum_{j=1}^{N}(\eta_{2j}\eta_{2j+1}\eta_{2j+2}\eta_{2j+3}\eta_{2j+4}\eta_{2j+5} - \eta_{2j-1}\eta_{2j}\eta_{2j+1}\eta_{2j+2}\eta_{2j+3}\eta_{2j+4})$$

generally

$$A_l = \sum_{j=1}^{N} \eta_{2j}\eta_{2j+1}\cdots\eta_{2j+2l-2},$$

$$A_{-l} = \sum_{j=1}^{N} \eta_{2j-2l-1}\eta_{2j-2l}\cdots\eta_{2j-1},$$

$$G_l = \frac{1}{2}\sum_{j=1}^{N}(\eta_{2j}\eta_{2j+1}\eta_{2j+2}\cdots\eta_{2j+2l-1} - \eta_{2j-1}\eta_{2j}\eta_{2j+1}\cdots\eta_{2j+2l-2})$$

Onsager algebra

$$[A_j, A_k] = 4G_{j-k}$$
$$[G_m, A_l] = 2A_{l+m} - 2A_{l-m}$$
$$[G_j, G_k] = 0$$

we also have

$$[A_j,[A_j,[A_j,A_k]]] = 16[A_j,A_k]$$
$$[A_j,[A_j,G_k]] = 16G_k$$

Summary of the formula

Find the series of operators

$$\eta_1 \quad \eta_2 \quad \eta_3 \quad \eta_4 \quad \eta_5 \quad \cdots$$

which satisfy

$$\eta_j \eta_k = -\eta_k \eta_j \quad (k = j \pm 1 : \text{when adjacent anticommute}), \quad \eta_j^2 = 1 \quad (k = j),$$
$$\eta_j \eta_k = \eta_k \eta_j \quad (|j-k| \geq 2 : \text{otherwise commute}).$$

Then the following Hamiltonian is solvable

$$-\beta\mathcal{H} = \sum_{j=1}^{N} K_j \eta_j$$

with the use of the transformation

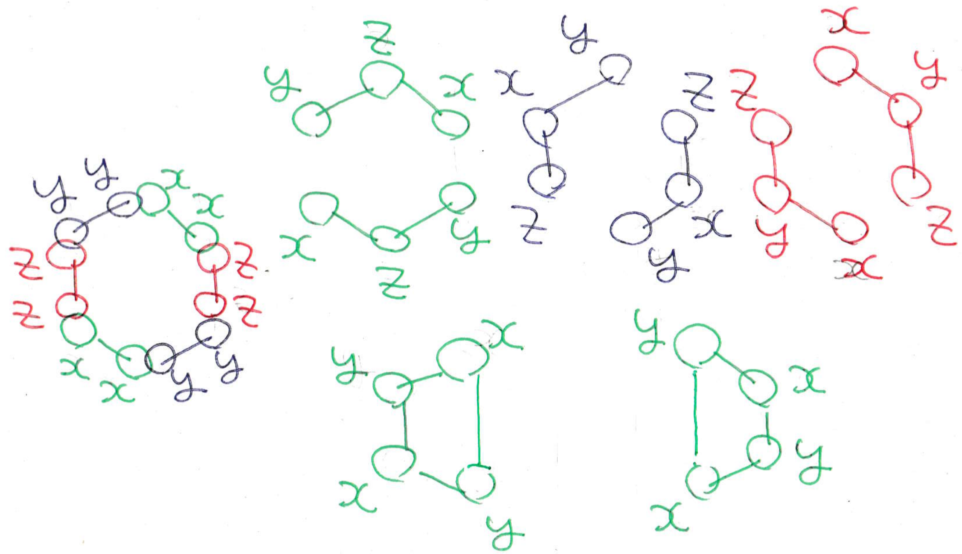
$$\varphi_j = \frac{1}{\sqrt{2}} e^{i\frac{\pi}{2}(j-1)} \eta_0 \eta_1 \eta_2 \cdots \eta_j.$$

Summary

- New fermionization method
  Examples（1-dim transv. Ising model, Kitaev model,cluster model,
  2-dim Ising model XY model）
- Infinite number of solvable models, with c=m/2
- Jordan-Wigner transformation is a special case

- Phase diagram of the 2-dim
  Kitaev model + Wen model
- excitations, Anyon

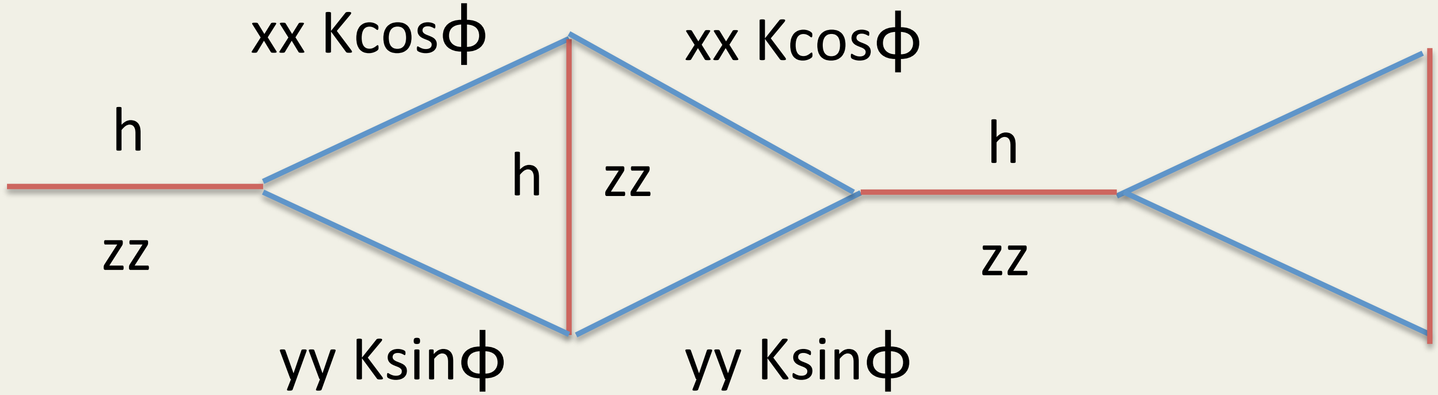
- realizations of the
  Onsager algebra



References

[1] K. Minami, J. Phys. Soc. Jpn. 85, 024003 (2016).
[2] K. Minami, Nuclear Physics B 925, 144 (2017).
[3] K. Minami, Nuclear Physics B (2019), in press.
[4] 「格子模型の数理物理」南 和彦, サイエンス社, 2014.
[5] http://www.math.nagoya-u.ac.jp/~minami/

Other systems



Series of operators

$$\eta_1 = \sigma^z_{j-3}\sigma^z_{j-2} \quad \eta_2 = \sigma^x_{j-2}\tilde{\sigma}^x_{j-1}\cos\phi + \sigma^y_{j-2}\sigma^y_{j-1}\sin\phi$$
$$\eta_3 = \tilde{\sigma}^z_{j-1}\sigma^z_{j-1} \quad \eta_4 = \tilde{\sigma}^x_{j-1}\sigma^x_{j}\cos\phi + \sigma^y_{j-1}\sigma^y_{j}\sin\phi$$
$$\eta_5 = \sigma^z_{j}\sigma^z_{j+1} \quad \eta_6 = \sigma^x_{j+1}\tilde{\sigma}^x_{j+2}\cos\phi + \sigma^y_{j+1}\sigma^y_{j+2}\sin\phi \quad \ldots$$

Commutation relationst

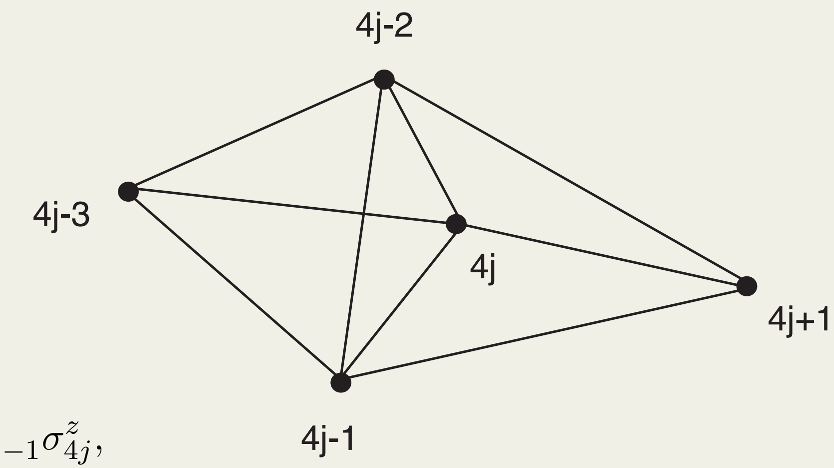
$$\eta_j\eta_k = -\eta_k\eta_j \quad (k = j \pm 1), \quad \eta_j\eta_k = \eta_k\eta_j \quad (|j-k| \geq 2), \quad \eta_j^2 = 1$$

Hamiltonian

$$-\beta\mathcal{H} = K\sum_{j=odd}^{N}\eta_j + h\sum_{j=even}^{N}\eta_j$$

Partition function

$$Z = \prod_{0<q<\pi}(e^{\frac{1}{2}\Lambda_q} + e^{-\frac{1}{2}\Lambda_q})^2 \quad \Lambda_q = 2\sqrt{K^2 + h^2 + 2Kh\cos q}$$



n-body products of the Pauli spin operators

$$\begin{aligned}\eta_{4j-3} &= \sigma^x_{4j-3},\\ \eta_{4j-2} &= \sigma^z_{4j-3}\sigma^z_{4j-2}\sigma^z_{4j-1}\sigma^z_{4j},\\ \eta_{4j-1} &= \sigma^x_{4j-2}\sigma^x_{4j-1}\sigma^x_{4j},\\ \eta_{4j} &= \sigma^z_{4j-2}\sigma^z_{4j-1}\sigma^z_{4j}\sigma^z_{4j+1}.\end{aligned}$$

These operators satisfy the condition.

Hamiltonian

$$-\beta\mathcal{H} = h\sum_{j=\mathrm{odd}}\eta_j + K\sum_{j=\mathrm{even}}\eta_j.$$

generally when we consider

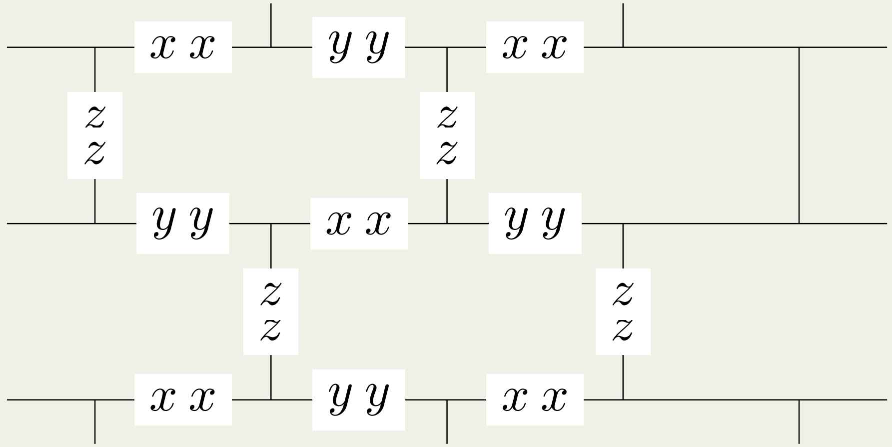
$$\eta_j = \mathcal{O}^L_j\mathcal{O}_j\mathcal{O}^R_j,$$

where $\mathcal{O}^L_j$, $\mathcal{O}_j$, $\mathcal{O}^R_j$ are products of Pauli operators which satisfy

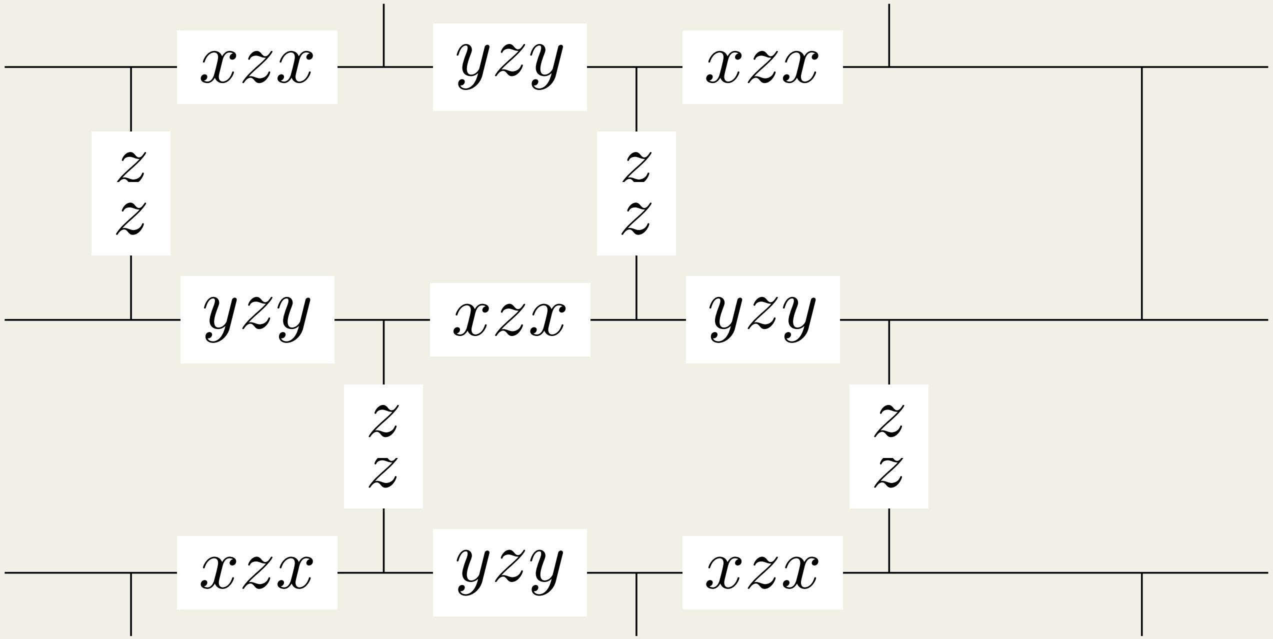
$$\{\mathcal{O}^R_{j-1}, \mathcal{O}^L_j\} = 0,$$

then η_j satisfy the condition.

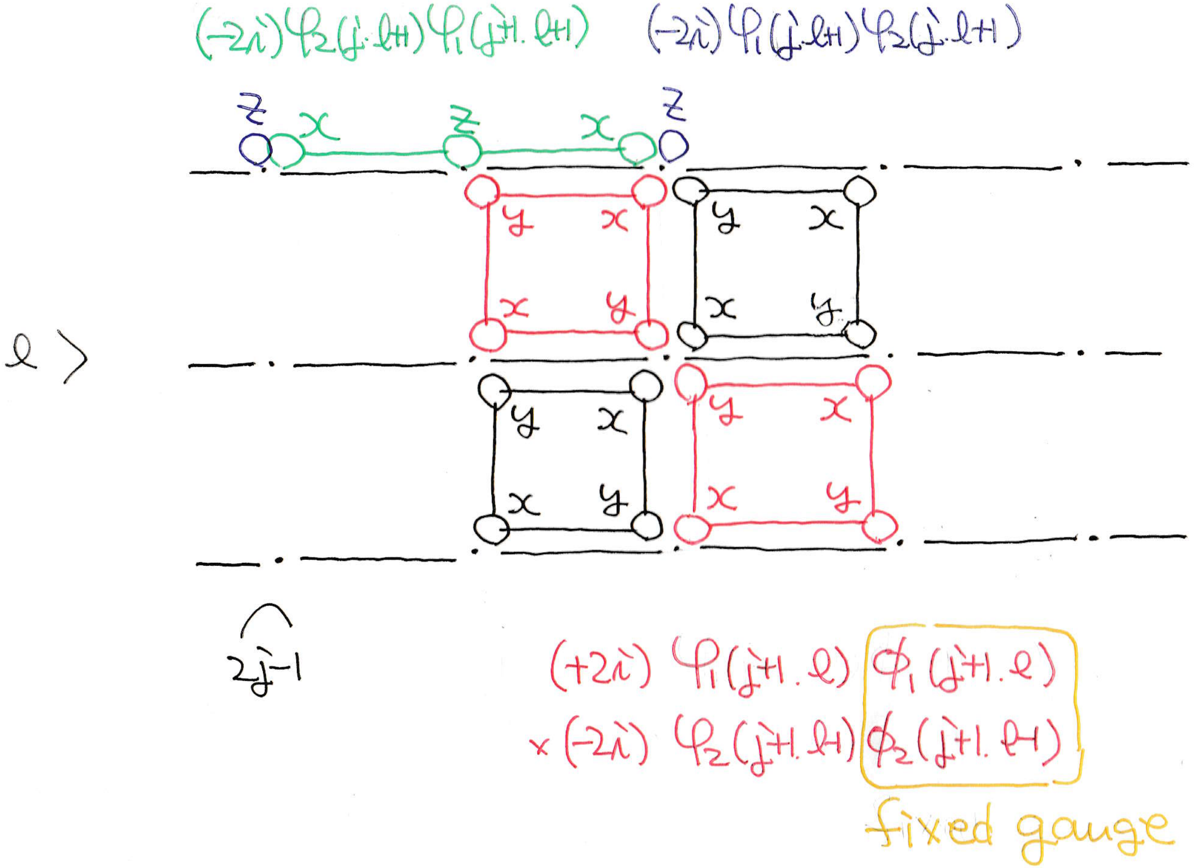
Kitaev model



Equivalent to the
Honeycomb-lattice Kitaev model



A two-dimensional model
hat cannnot be solved by the Jordan-Wigner transformation



拡張されたcluster模型

cluster 模型

$$-\beta\mathcal{H} = K_1\sum_{j=1}^{N}\sigma_j^x\sigma_{j+1}^z\sigma_{j+2}^x + (K_2\sum_{j=1}^{N}\sigma_j^y\sigma_{j+1}^z\sigma_{j+2}^y + K_3\sum_{j=1}^{N}\sigma_j^y\sigma_{j+1}^y)$$

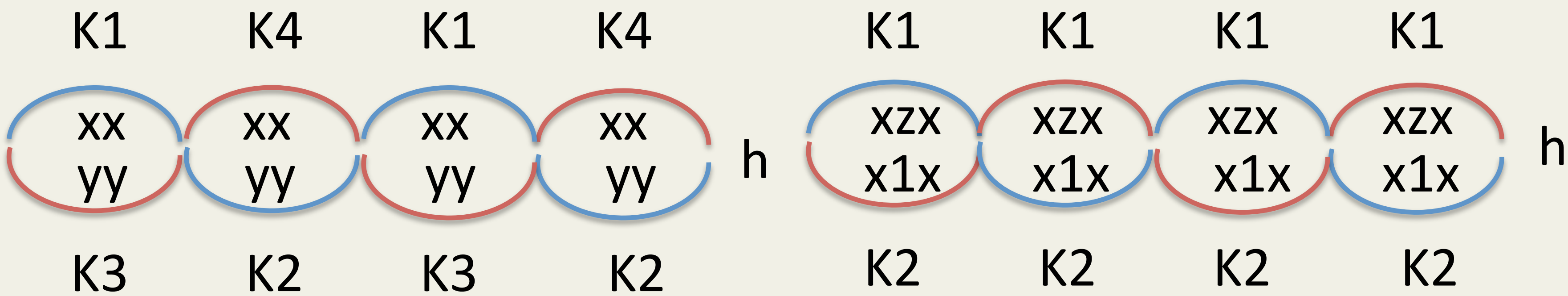
一般化された cluster 模型 1

$$-\beta\mathcal{H} = K_1\sum_{j=1}^{N}\sigma_j^x\sigma_{j+1}^z\sigma_{j+2}^x + K_2\sum_{j=1}^{N}\sigma_j^x\,\mathbf{1}_{j+1}^z\sigma_{j+2}^x$$

変換

$$\varphi_1(j) = \begin{cases} \frac{1}{\sqrt{2}}\ \mathbf{1}_1\,\sigma_2^z\,\mathbf{1}_3\,\sigma_4^z\,\mathbf{1}_5\,\sigma_6^z\cdots\cdots\mathbf{1}_j\,\sigma_{j+1}^y\,\sigma_{j+2}^x & j=\mathrm{odd} \\ \frac{1}{\sqrt{2}}\ \mathbf{1}_1\,\sigma_2^z\,\mathbf{1}_3\,\sigma_4^z\,\mathbf{1}_5\,\sigma_6^z\cdots\cdots\sigma_j^z\,\sigma_{j+1}^x\,\sigma_{j+2}^x & j=\mathrm{even} \end{cases}$$

$$\varphi_2(j) = \begin{cases} \frac{1}{\sqrt{2}}\ \sigma_1^z\,\mathbf{1}_2\,\sigma_3^z\,\mathbf{1}_4\,\sigma_5^z\,\mathbf{1}_6\cdots\cdots\sigma_j^z\,\sigma_{j+1}^x\,\sigma_{j+2}^x & j=\mathrm{odd} \\ \frac{1}{\sqrt{2}}\ \sigma_1^z\,\mathbf{1}_2\,\sigma_3^z\,\mathbf{1}_4\,\sigma_5^z\,\mathbf{1}_6\cdots\cdots\mathbf{1}_j\,\sigma_{j+1}^y\,\sigma_{j+2}^x & j=\mathrm{even} \end{cases}$$



cluster 模型

$$-\beta\mathcal{H} = K_1 \sum_{j=1}^{N} \sigma_j^x \sigma_{j+1}^z \sigma_{j+2}^x + (K_2 \sum_{j=1}^{N} \sigma_j^y \sigma_{j+1}^z \sigma_{j+2}^y + K_3 \sum_{j=1}^{N} \sigma_j^y \sigma_{j+1}^y)$$

一般化された cluster 模型 1

$$-\beta\mathcal{H} = K_1 \sum_{j=1}^{N} \sigma_j^x \sigma_{j+1}^z \sigma_{j+2}^x + K_2 \sum_{j=1}^{N} \sigma_j^x \, \mathbf{1}_{j+1}^z \sigma_{j+2}^x$$

結合した transverse Ising 模型

$$\begin{aligned} -\beta\mathcal{H} &= (K_1 \sum_{j=odd} \sigma_j^x \, \mathbf{1}_{j+1}^z \, \sigma_{j+l+1}^x + K_2 \sum_{j=odd} \sigma_j^z) \\ &+ (K_1 \sum_{j=even} \sigma_j^x \, \mathbf{1}_{j+1}^z \, \sigma_{j+l+1}^x + K_2 \sum_{j=even} \sigma_j^z) \\ &+ K_3 \sum_{j=1}^{N} \sigma_1^z \sigma_2^z \sigma_3^z \cdots \sigma_j^z \end{aligned}$$

Hamiltonian

$$-\beta\mathcal{H} = K_{-m}\sum_{j=1}^{N}\varphi_2(j)\varphi_1(j+\kappa-m) + K_0\sum_{j=1}^{N}\varphi_2(j)\varphi_1(j+\kappa) + K_m\sum_{j=1}^{N}\varphi_2(j)\varphi_1(j+\kappa+m).$$

In this case

$$\begin{aligned}
f(p) &= \frac{L^\dagger}{\sqrt{LL^\dagger}}e^{ip\kappa}, \qquad L = \alpha_{-m}e^{ip(\kappa-m)} + e^{ip\kappa} + \alpha_m e^{ip(\kappa+m)} \\
\alpha_m &= K_m/K_0,\ \ \alpha_{-m} = K_{-m}/K_0,\ \ a_m = \alpha_m t > a_{-m} = \alpha_{-m} t \\
I_\infty &= \lim_{n\to\infty}\frac{\det M_n}{\mu^n} = \exp(\sum_{n=1}^{\infty} n g_n g_{-n}) \\
&= \begin{cases} \pm[(1-a_m^2)(1-a_{-m}^2)/(1-a_m a_{-m})]^{|m|/4} & |a_m| < 1 \\ 0 & |a_m| > 1 \end{cases}
\end{aligned}$$

Exponent=m/4

Ex. transverse Ising model

$$\langle\sigma_j^x\rangle_0^2 = \lim_{n\to\infty}\langle\sigma_j^x\sigma_{j+n}^x\rangle_0 \simeq \pm[(1-K_m/K_0)(1+K_m/K_0)]^{1/4} \qquad \beta = 1/8.$$

相関関数

$$
\begin{aligned}
\langle \sigma_j^z \rangle &= \langle \sigma_1^z \cdots \sigma_{j-1}^z \cdot \sigma_1^z \cdots \sigma_j^z \rangle = \rho_{j-1 j} \\
\langle \sigma_2^z \cdots \sigma_{n+1}^z \rangle &= \langle \sigma_1^z \cdot \sigma_1^z \cdots \sigma_{n+1}^z \rangle = \rho_{1 n+1}
\end{aligned}
$$

ただし

$$
\begin{aligned}
\rho_{lm} &= \frac{1}{4} \begin{vmatrix} G_{ll} & G_{lm} \\ G_{ml} & G_{mm} \end{vmatrix} \\
G_{lm}(\beta) &= -\frac{1}{2} \frac{K_1}{\sqrt{K_1^2 + K_2^2}} L_{r+1} + \frac{1}{2} \frac{K_2}{\sqrt{K_1^2 + K_2^2}} L_{r-1}, \qquad r = |l - m| \\
L_r &= \frac{2}{\pi} \int_0^{\pi/2} \frac{1}{\Lambda_k} \cos k(r-1) \tanh \frac{1}{2} \beta \Lambda_k \, dk
\end{aligned}
$$