Spacetimes in the canonical tensor model through data analysis techniques

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Based on a collaboration with D.Obster and T.Kawano
Phys.Rev. D97 (2018), 124061 (arXiv:1805.04800)

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The problem of quantum gravity

It is hard to quantize general relativity — Difficult by the standard field theoretical renormalization procedure: Quantum fluctuations in short-distances diverge too seriously.

An idea for a way out : Leave from the conventional description of spacetimes, and use new variables to describe them.

Emergent spacetime : Spacetime and general relativity (and other field theories) are emergent phenomena valid only in the scales much larger than the Planck scale.

Quantum gravity : A theory / model, which has such new variables, is quantizable, and generates macroscopic spacetimes.

Tensor models

Generalization of the matrix models to tensors, based on that the matrix models successfully describe 2-dim quantum gravity.

J,Ambjorn, B.Durhuus, T.Jonsson, Mod.Phys.Lett. A6 (1991) 1133-1146
NS, Mod.Phys.Lett. A6 (1991) 2613-2624

$$M_{ab} \rightarrow T_{a_1 \ldots a_D}$$

$$\int \prod_{a_1,a_2=1}^{N} dM_{a_1 a_2} \exp[-S(M)] \rightarrow \int \prod_{\{a_i\}}^{N} dT_{a_1 a_2 \ldots a_D} \exp[-S(T)]$$

Emergent space :

Feynman diagrams of tensor models ⟷ Simplicial spaces
Dual

e.g. D=3

However, the dominant graphs in large N limit of (colored) tensor models are too singular to be considered as macroscopic spaces — they are branched polymers or other non-smooth objects.

R. Gurau and J. P. Ryan, Annales Henri Poincare 15 no. 11, (2014) 2085–2131
V. Bonzom, R. Gurau, A. Riello, and V. Rivasseau, Nucl. Phys. B853 (2011) 174–195
L. Lionni and J. Thurigen, arXiv:1707.08931

So far, from the view point of quantum gravity, it seems hard to obtain emergence of macroscopic spaces from the tensor models.

Another option of tensor model

NS, Int.J.Mod.Phys. A27 (2012) 1250020, arXiv:1111.2790

A tensor model in Hamilton formalism (or canonical formalism).

Dynamical variables: A canonical conjugate pair of tensors (symmetric, real, three indices)

Classical case: $\{Q_{abc}, P_{def}\} = \delta_{ad}\delta_{be}\delta_{cf} + \cdots$

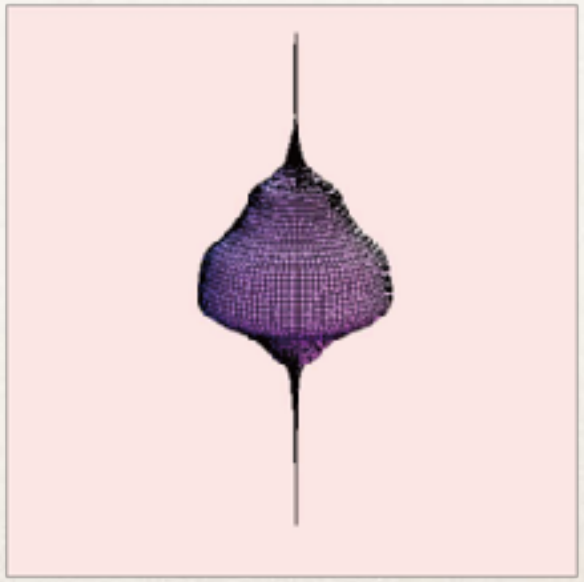
Quantum case: $[\hat{Q}_{abc}, \hat{P}_{def}] = i\,\delta_{ad}\delta_{be}\delta_{cf} + \cdots$

$a, b, \ldots = 1,2,\ldots,N$

We named the model Canonical Tensor Model (CTM).

Inspired by the success of Causal Dynamical Triangulation (CDT)
— Emergence of macroscopic spaces.

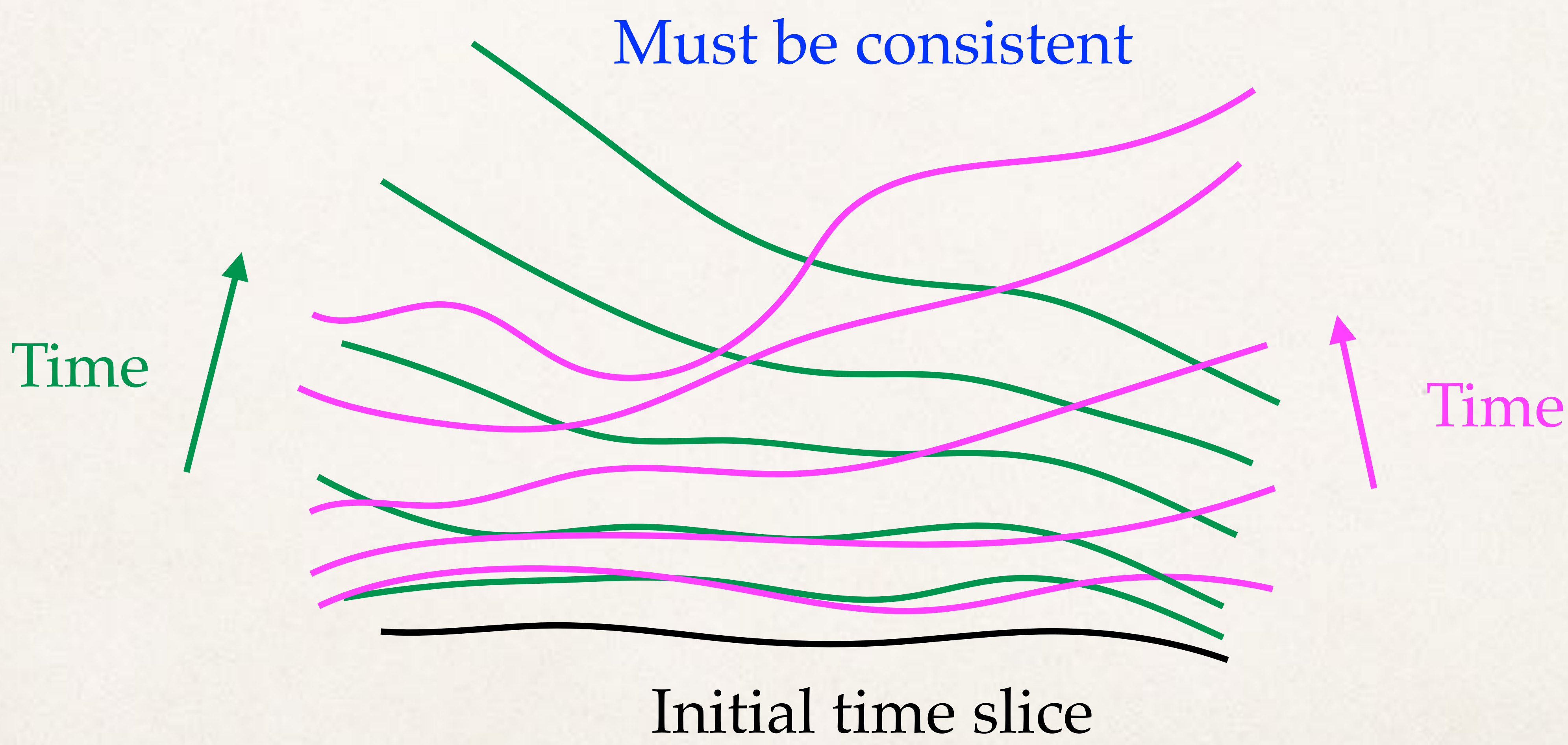
Causal structure seems essentially important in quantum gravity.



Taken from J. Ambjorn, J. Jurkiewicz, R. Loll, Contemp.Phys. 47 (2006) 103-117

To describe gravity, Hamiltonian formalism needs special treatment.

One has to incorporate "general covariance" to assure the independence of physics from spacetime foliations.



A well-known way to do: Hamiltonian is a linear combination of first-class constraints, as in the Hamilton (ADM) formalism of general relativity.

$$\mathscr{H}(q,p,N) = N^i H_i(q,p)$$

$$\{H_i(q,p), H_j(q,p)\} = f_{ij}^k(q,p)\, H_k(q,p)$$

N_i : Arbitrary parameters determining the way of time evolutions.

First-class constraints form a closed poisson algebra.

In the quantum case, similar closed structure must exist.

$$q,p \rightarrow \hat{q},\hat{p} \qquad \{\ ,\ \} \rightarrow \frac{1}{i}[\ ,\ ]$$

In tensor model, the closure condition puts strong constraints, and there exists only a unique choice under physically reasonable assumptions.

NS, Int.J.Mod.Phys. A27 (2012) 1250096, arXiv:1203.0421

$$\mathscr{H} = N_a H_a + N_{ab} J_{ab} \qquad (a, b, \ldots = 1,2,\ldots,N)$$

"Hamiltonian" constraints

$$H_a = \frac{1}{2}\left(P_{abc}P_{bde}Q_{cde} - \lambda\, Q_{abb}\right) \qquad \lambda = 0, \pm 1$$

"Momentum" constraints

$$J_{ab} = \frac{1}{4}\left(Q_{acd}P_{bcd} - Q_{bcd}P_{acd}\right) \qquad \text{(SO(N) generators)}$$

Quantum case:

$$\hat{H}_a = \frac{1}{2}\left(\hat{P}_{abc}\hat{P}_{bde}\hat{Q}_{cde} - \lambda\, \hat{Q}_{abb} + \underline{i\,\lambda_H\, \hat{P}_{abb}}\right)$$

↑ From normal ordering

$$\lambda_H = \frac{(N+2)(N+3)}{2}$$

The Poisson algebra of the constraints :

$$\{H(n_1), H(n_2)\} = J\left([\tilde{n}_1, \tilde{n}_2] + 2\lambda\, n_1 \wedge n_2\right)$$

$$\{J(m), H(n)\} = H(mn)$$

$$\{J(m_1), J(m_2)\} = J\left([m_1, m_2]\right)$$

Non-linearity

$$\tilde{n}_{bc} \equiv n_a P_{abc}$$

$$H(n) \equiv n_a H_a \quad J(m) \equiv m_{ab} J_{ab}$$

$$(n_1 \wedge n_2)_{ab} \equiv n_{1a} n_{2b} - n_{1b} n_{2a}$$

Very similar to the structure in ADM formalism of general relativity.

The same algebraic structure holds also in the quantum case.

Main previous results on CTM

(1) Connections between classical CTM and general relativity (GR)

- The N=1 case agrees with the mini-superspace treatment of GR.

  NS, Y.Sato, Phys.Lett. B732 (2014) 32-35, arXiv:1401.2062

- The classical e.o.m. of CTM, H.Chen, NS, Y.Sato, Phys.Rev. D95 (2017), 066008, arXiv:1609.01946

$$\frac{d}{dt}P_{abc} = \{P_{abc}, \mathscr{H}\} = -N_d P_{de(a}P_{bc)e} + N_{d(a}P_{bc)} \quad (\ldots): \text{symmetrization}$$

agrees with that in the Hamilton-Jacobi formalism of a GR system with the following Liouville-type action

$$S = \int d^{D+1}x \left(2R - \frac{1}{2}(\nabla\phi)^2 - e^{-\alpha\phi} + \text{ higher derivatives/spins} \right)$$

$$\alpha = \sqrt{(6-D)/8(D-1)}$$

in a formal continuum limit,

$$P_{abc} \to P_{xyz} \qquad a, b, c \in \mathbb{N} \longrightarrow x, y, x \in \mathbb{R}^D,\ x \sim y \sim z$$

(2) In the quantum case, there exists a physical state which can be solved exactly.

G.Narain, NS, Y.Sato, JHEP 1501 (2015) 010, arXiv:1410.2683

Physical state condition : $\hat{H}_a|\Psi\rangle = \hat{J}_{ab}|\Psi\rangle = 0$

In terms of a wave function in P_{abc}, this gives a system of first-order partial differential equations :

$$\left(P_{abc}P_{bde}\frac{\partial}{P_{cde}} - \lambda\frac{\partial}{P_{abb}} + i\,\lambda_H P_{abb} \right)\Psi(P) = 0$$

$$\left(P_{acd}\frac{\partial}{\partial P_{bcd}} - P_{bcd}\frac{\partial}{\partial P_{acd}} \right)\Psi(P) = 0$$

A systematic exact solution :

$$\Psi(P) = \psi(P)^{\frac{\lambda_H}{2}}$$

$$\psi(P) = \int_C \prod_{a=1}^{N} d\phi_a d\tilde{\phi}\, \exp\left(P_{abc}\phi_a\phi_b\phi_c + \phi_a\phi_a\tilde{\phi} - \frac{4}{27\lambda}\tilde{\phi}^3\right)$$

- The contour C must be taken appropriately to make the integral convergent. Picard-Lefschetz theory may be applicable.
- A sort of generalization of the Airy function to the case of multiple integration variables.
- Looks like a sort of bosonic SYK model, if random P_{abc} is considered.

The "physical" choice of the contour :

$$\psi(P) = \lim_{\epsilon \to +0} \int_{R^{N+1}} \prod_{a=1}^{N} d\phi_a d\tilde{\phi} \, \exp\left(i \left(P_{abc}\phi_a\phi_b\phi_c + \phi_a\phi_a\tilde{\phi} - \frac{4}{27\lambda}\tilde{\phi}^3 \right) - \epsilon \, \phi^2 \right)$$

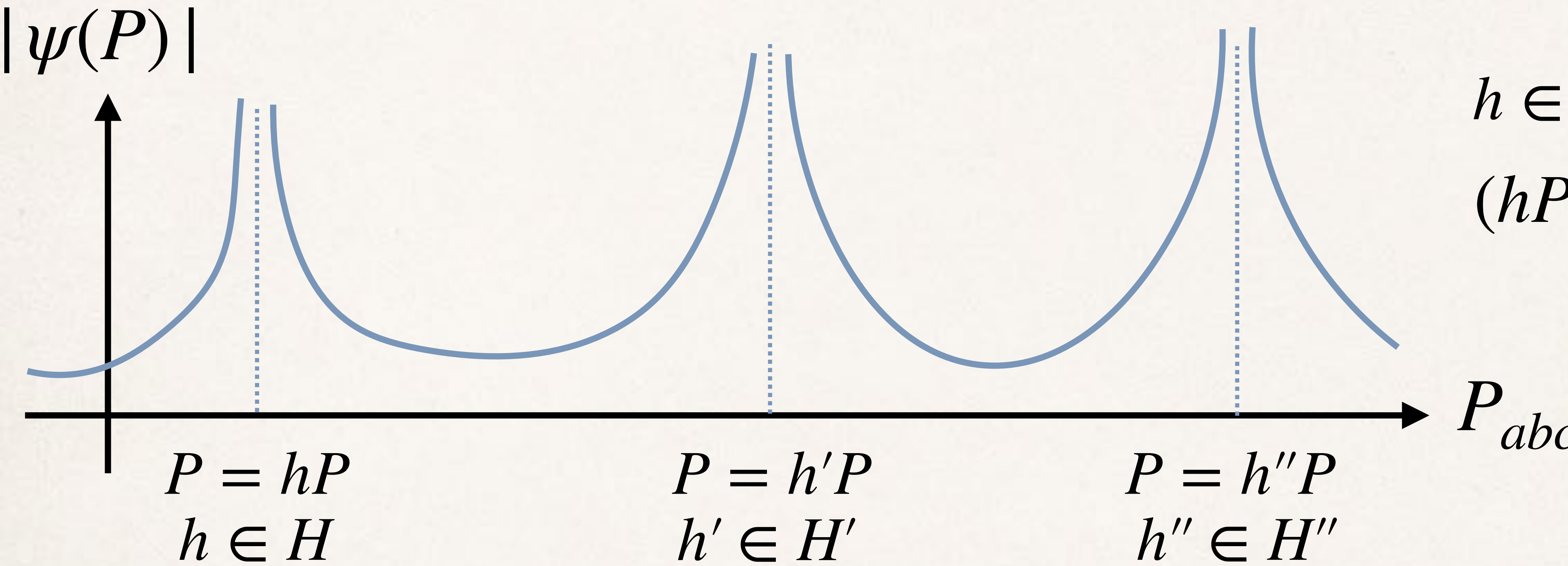
Feynman prescription

- $\psi(P)$ is finite for general real P_{abc} except for some singular loci. The singular loci can be obtained by solving the equations for critical points.

- In place of Feynman prescription, one may define the integral with small deformation of the contour from real.

- Singular loci are tightly related to Lie group symmetries. Large peaks exist at P_{abc} invariant under Lie groups.

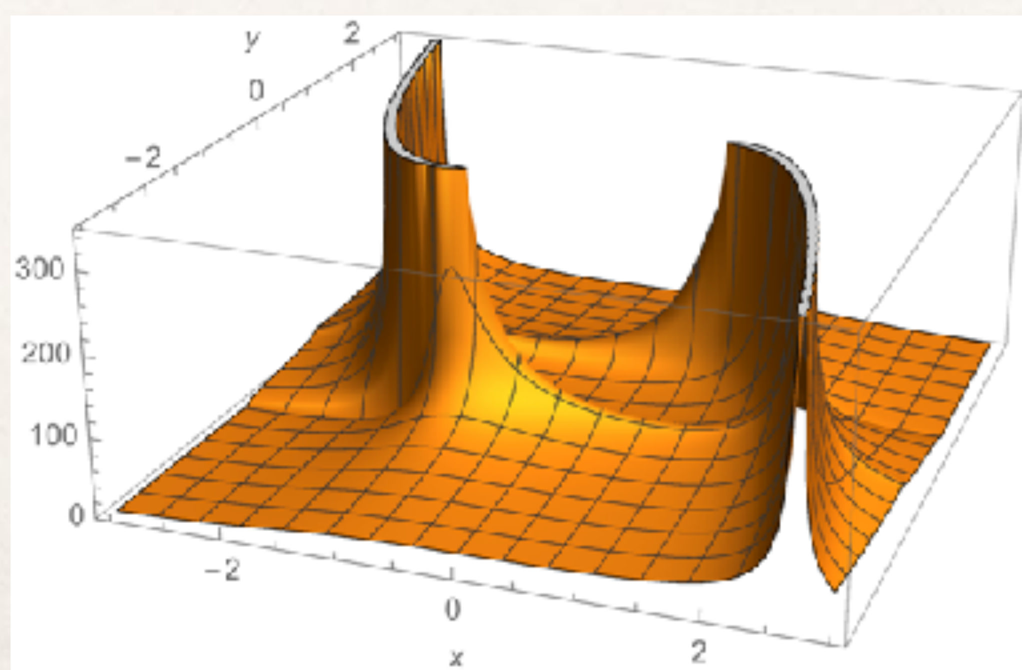
D.Obster, NS, PTEP 2018 (2018), 043A01, arXiv:1710.07449;
Eur.Phys.J. C77 (2017) no.11, 783, arXiv:1704.02113



$h \in H$

$(hP)_{abc} \equiv h_{(a}{}^{d} P_{bc)d}$

e.g. SO(2,1)



The question in this talk

Is the canonical tensor model a theory of spacetime ?

Is there a way to regard P_{abc} = a space (spacetime) ?

If answered,

- Classical EOM of P_{abc} = Time evolution of a space, namely, spacetime.
- P_{abc} at the peaks of the wave function

  ||

  Quantum mechanically favored spacetimes with Lie group symmetries.

  (Assuming the peaks are integrable.)

The answer in this talk

T.Kawano, D.Obster, NS, Phys.Rev. D97 (2018), 124061 (arXiv:1805.04800)

P_{abc} = Data in data analysis

↓

Geometric interpretation

J. M. Landsberg, *"Tensors: Geometry and Applications"*,
American Mathematical Society, Providence, 2012.

G. Carlsson, *"Topology and data"*,
Bulletin (New Series) of the American Mathematical Society, Vol 46,
Number 2, April 2009, Pages 255–308

More explicitly :

Given P_{abc}

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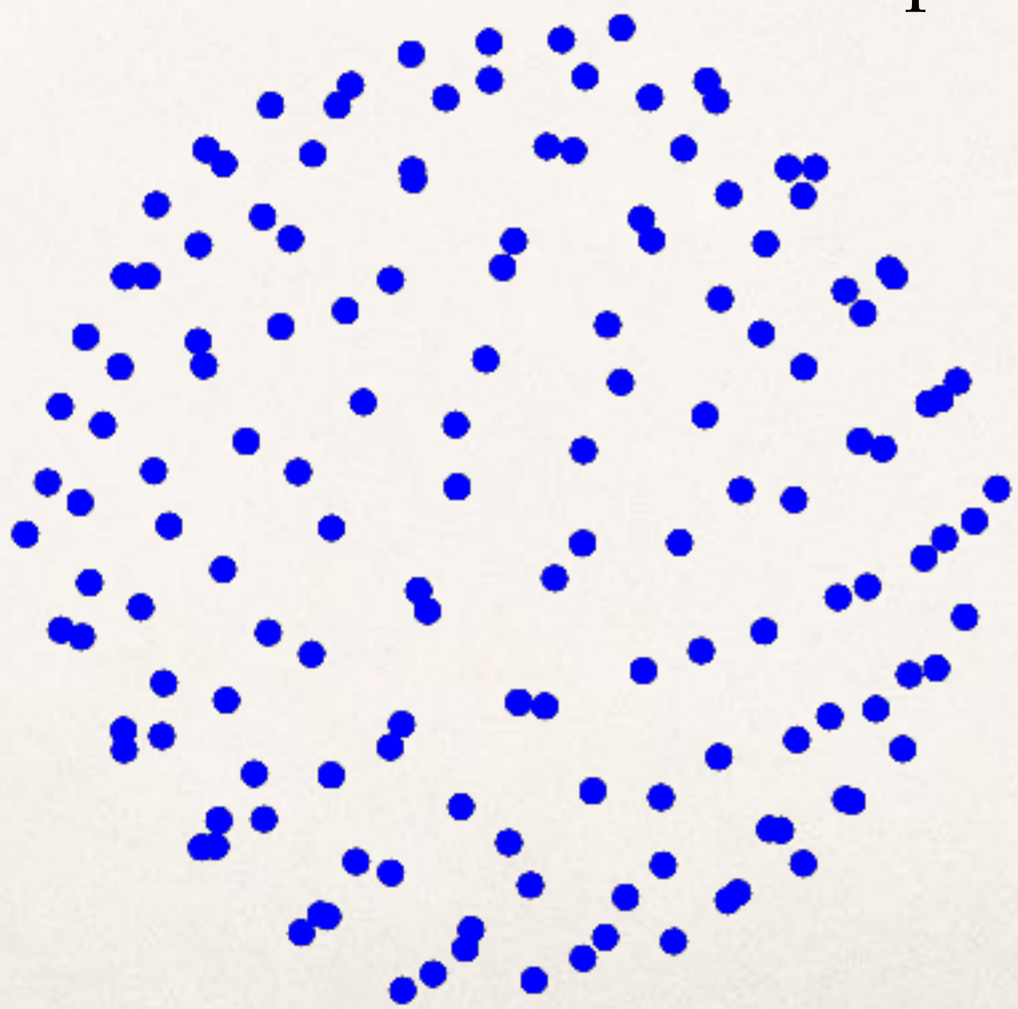
Given P_{abc}

↓

Tensor-rank decomposition → Points

Point cloud in N-dim. space

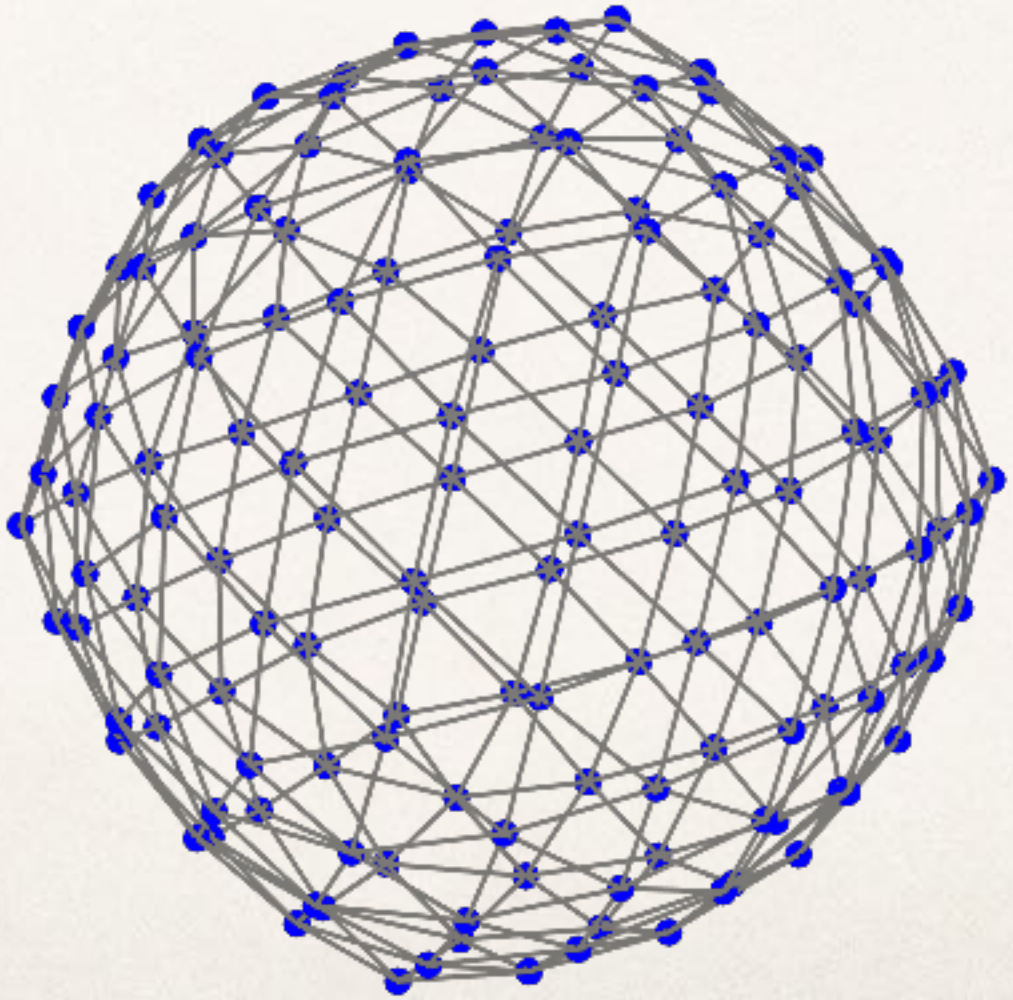


More explicitly :

Given P_{abc}

↓

Tensor-rank decomposition → Points & Neighborhoods



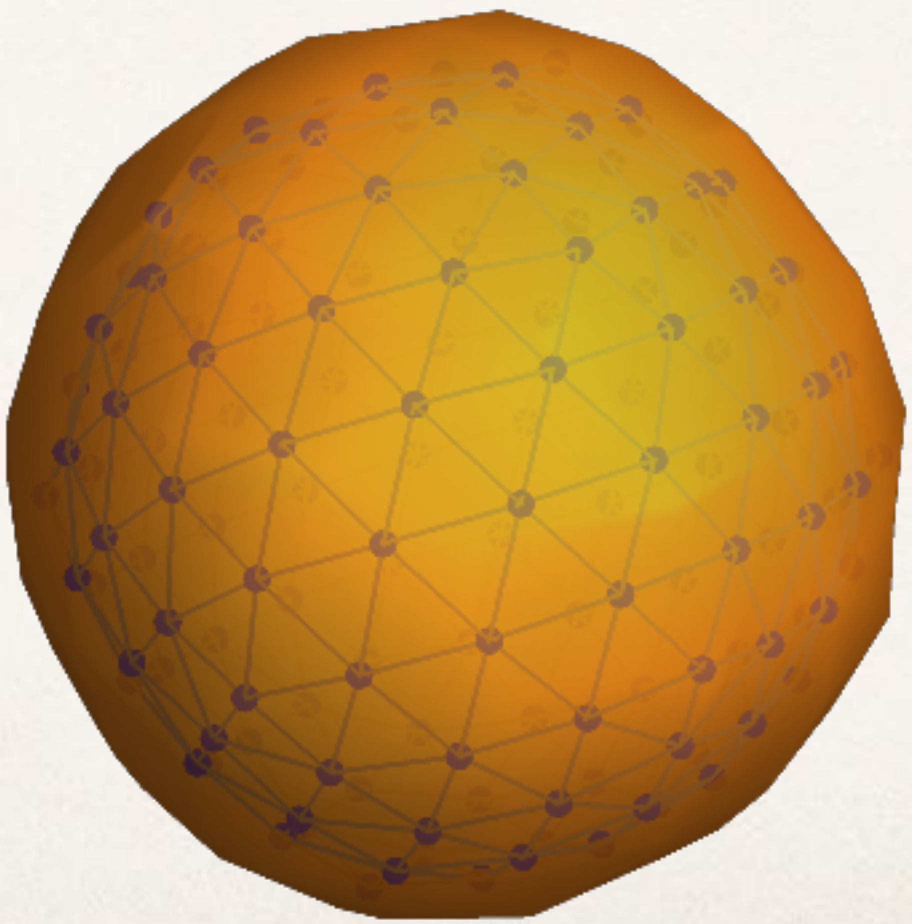
More explicitly :

Given P_{abc}

↓

Tensor-rank decomposition → Points & Neighborhoods

→ (Persistent homology → Topology



More explicitly :

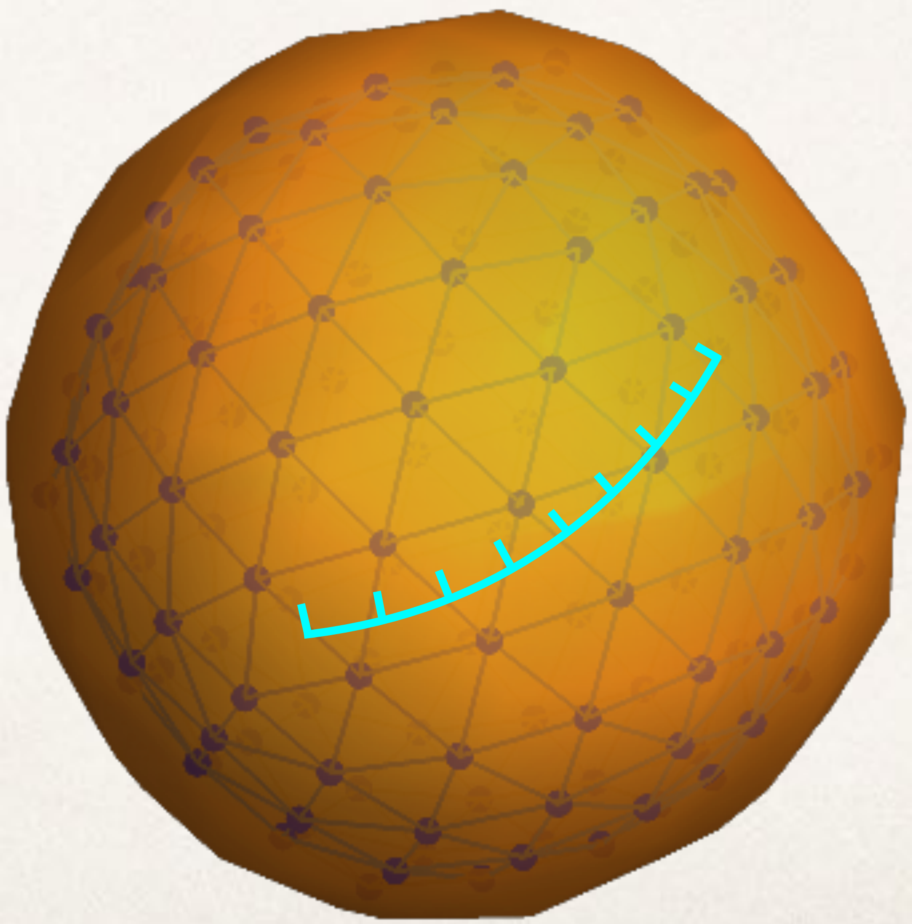
Given P_{abc}

↓

Tensor-rank decomposition → Points & Neighborhoods

→ (Persistent homology → Topology

Laplacian → Geometry)



Real symmetric tensor-rank decomposition

$$P_{abc} = \sum_{r=1}^{R} v_a^r v_b^r v_c^r \qquad a, b, c = 1,2,\ldots,N$$

$$v^r \in \mathbb{R}^N$$

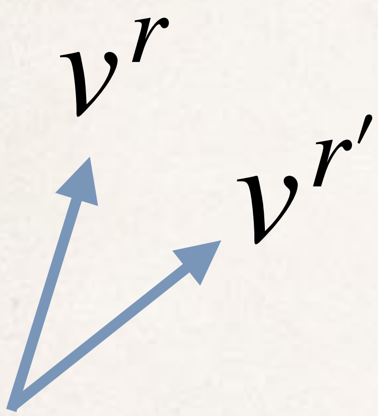
The smallest R is called the rank of a tensor.

Each summand $v_a^r v_b^r v_c^r$ is a rank-one tensor, which is the simplest tensor.

We regard each v^r $(r = 1,2,\ldots,R)$ represents a point. Point cloud.

The rank R is the total number of points forming a space.

It is natural to introduce the notion of neighborhood by inner products between vectors representing points.



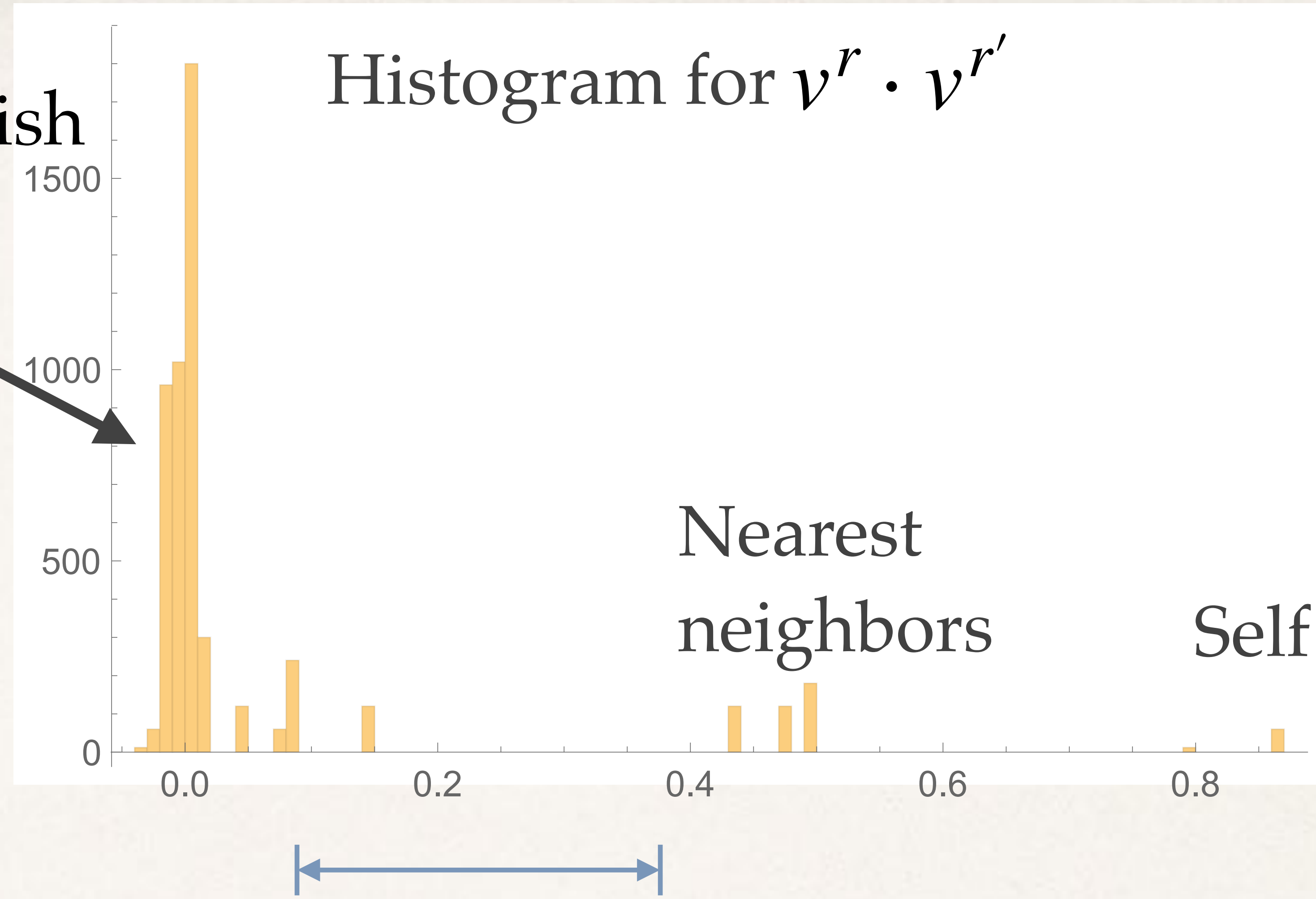
$$|v^r \cdot v^{r'}| > c \Longleftrightarrow \text{The points } r \text{ and } r' \text{ are neighbors.}$$

$$(v^r \cdot v^{r'} \equiv v^r_a v^{r'}_a)$$

The choice of cut-off c will introduce arbitrariness.
However, in our examples, there exist reasonable ranges.

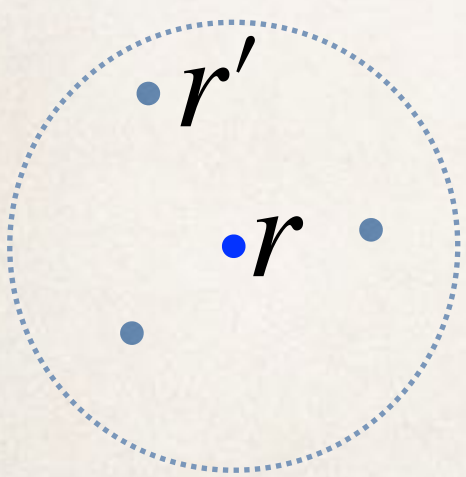
Most of them nearly vanish

$$v^r \cdot v^{r'} \sim 0$$



Reasonable range of c

$$|v^r \cdot v^{r'}| > c$$

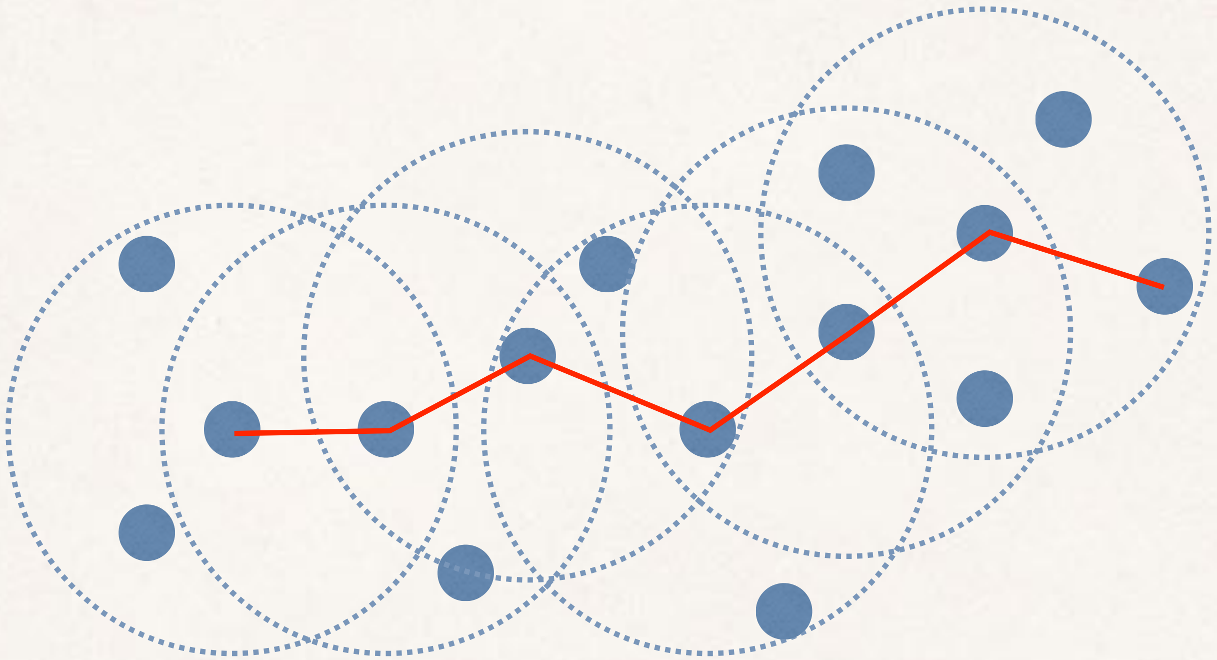


There are only a few neighboring points around each point: Locality is holding.

The other points cannot be seen from the point r.

Through this neighborhood relation, one can define the adjacency distance between points.

E.g.

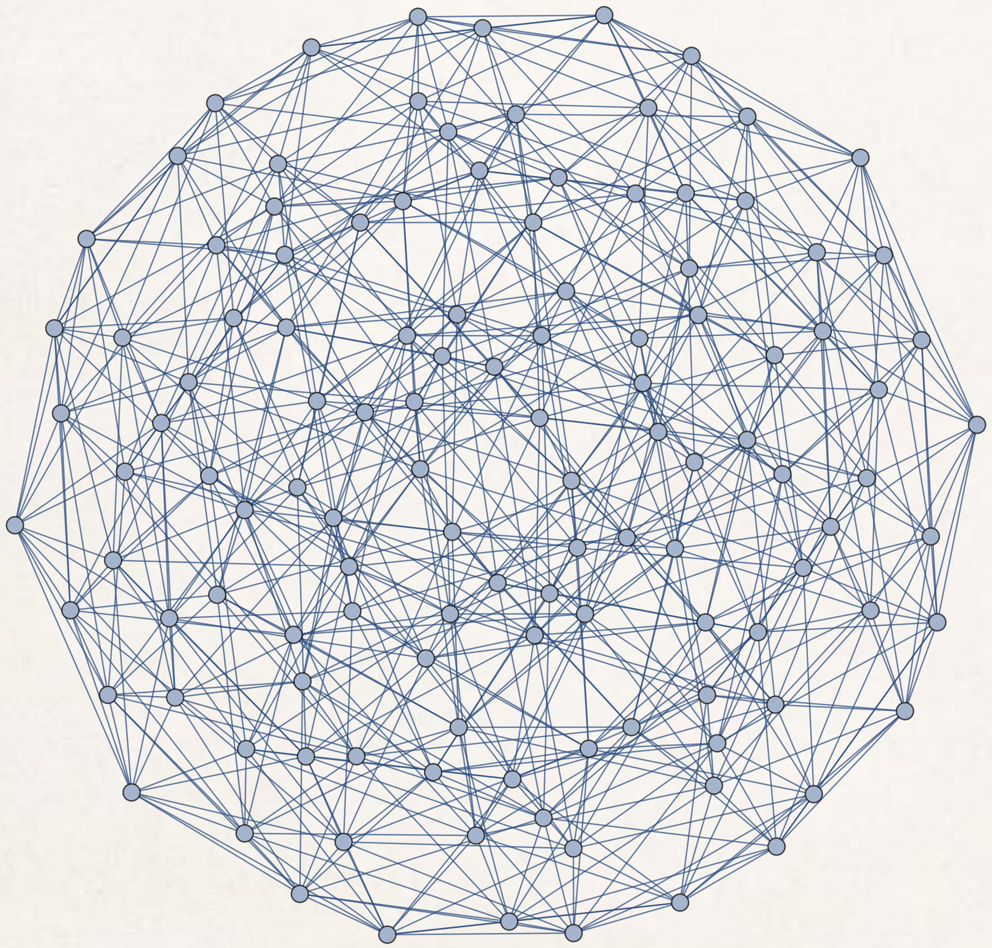


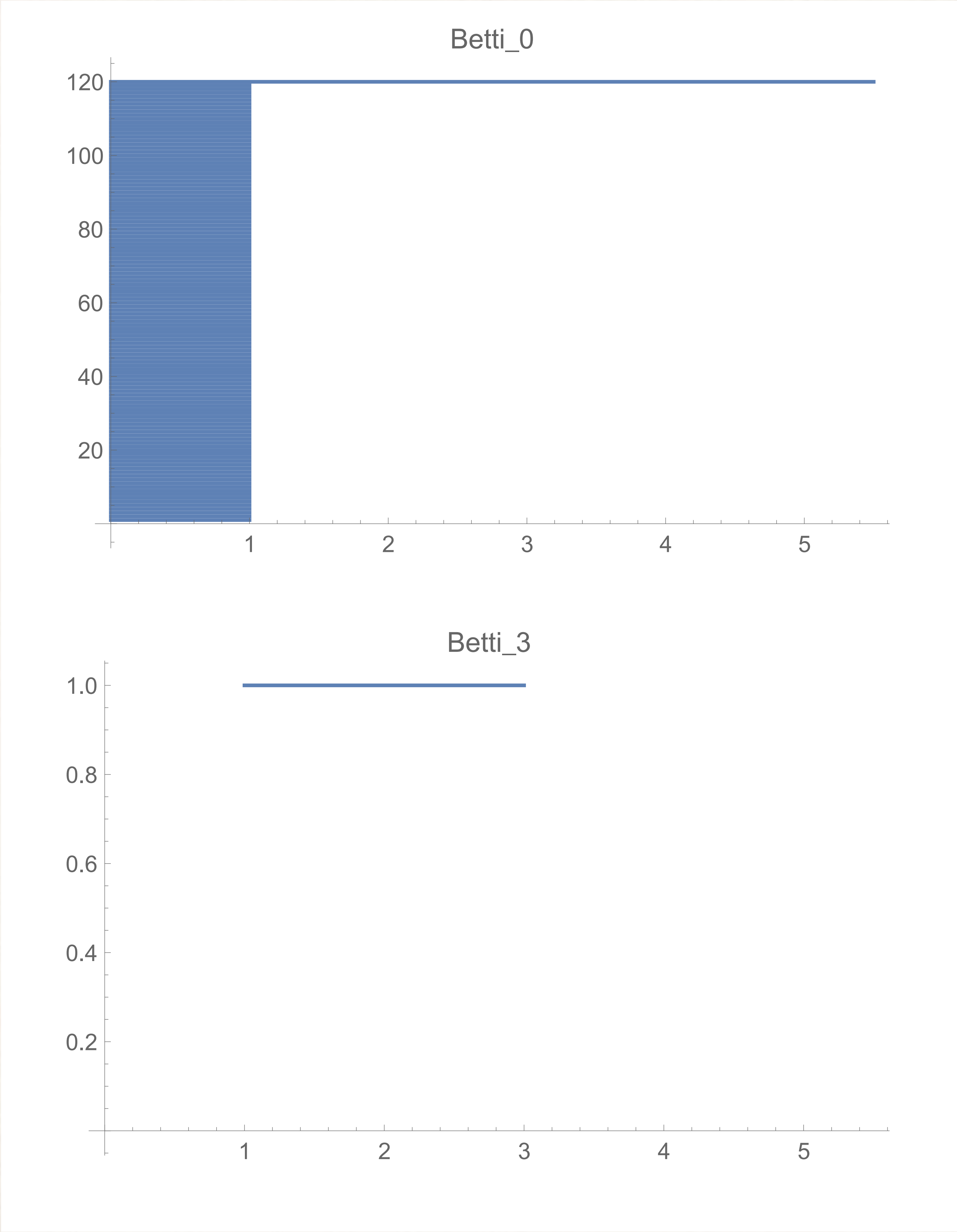
$d_{\mathrm{adj}}=6$

This distance is convenient to capture rough geometric properties.

Adjacency distance can be used to extract persistent homology.

Is this S^3 ?





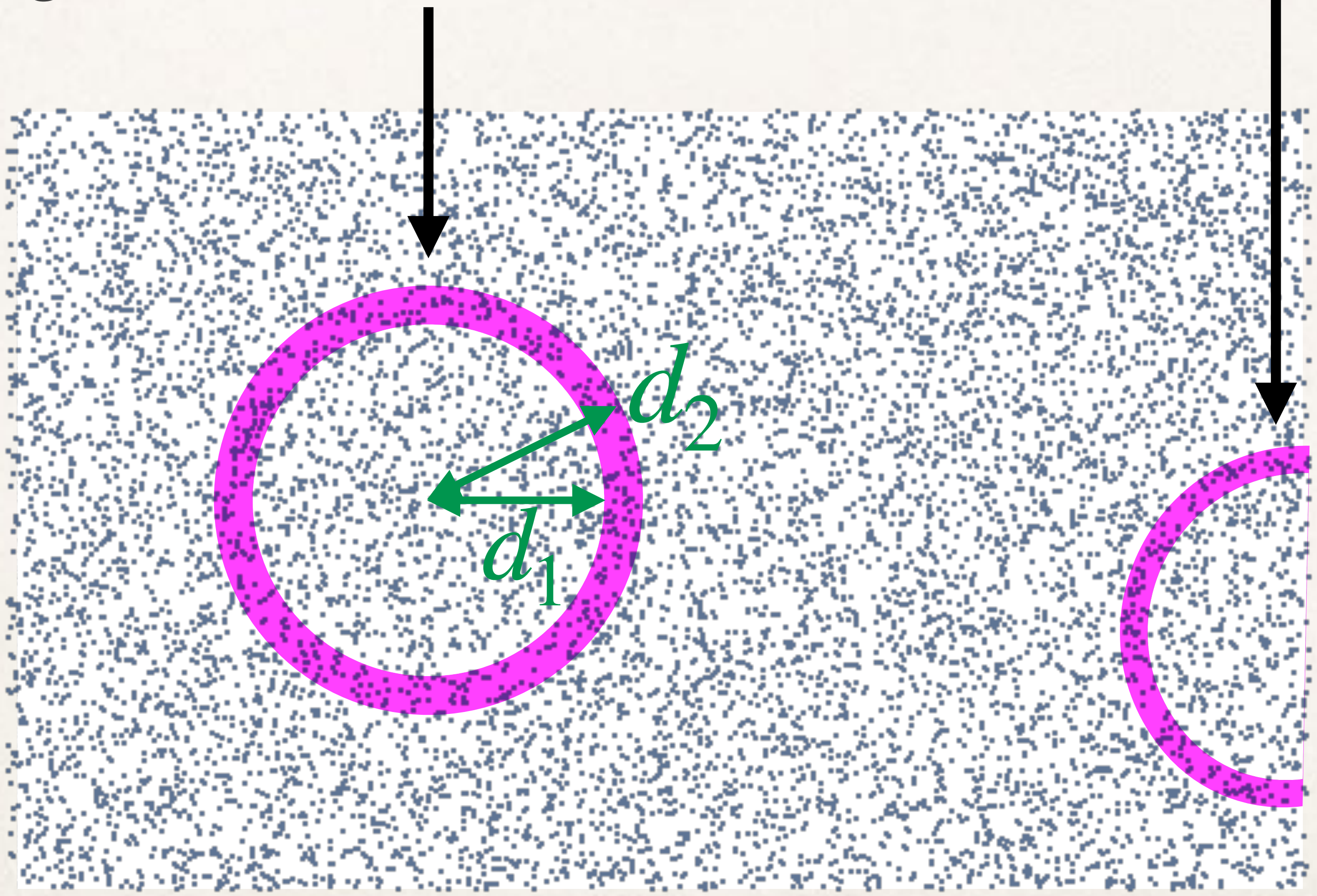
Computed by Ripser.

Other useful applications of persistent homology:

- Determine the topological dimension D
- Distinguish boundaries

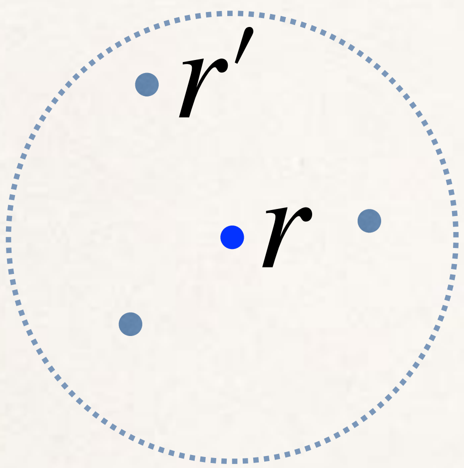
$\text{Region}(d_1 < d < d_2) \sim B^{D-1}$

$\text{Region}(d_1 < d < d_2) \sim S^{D-1}$



More detailed geometries — Metric and a scalar field

Tensor P_{abc} ⟶ Points : $v^r \in \mathbb{R}^N$ $(r = 1, 2, \ldots, R)$



Very roughly,

$|v^r|$ ⤳ A scalar field

$v^r \cdot v^{r'}$ ⤳ A metric field

Larger, shorter distance and vice versa.

In fact, the formal continuum limit of the canonical tensor model contains a scalar field, a metric field, and higher spin fields.

Not yet identified.

Higher moments of v^r ?

From the canonical tensor model, the following K has a good motivation as a Laplace operator. From K, one can deduce various geometric quantities.

(eg. Virtual diffusion process $d\rho/dt = K\rho$)

$$K_{rr'} = \beta(r)^{-1}\beta(r')^{-1} w^r \cdot w^{r'} - \delta_{rr'}$$

$$(r, r' = 1,2,\ldots,R)$$

↑

$$P_{abc} = \sum_{r=1}^{R} \frac{1}{\beta(r)^2} w_a^r w_b^r w_c^r$$: Tensor-rank decomposition

$$\sum_{r'=1}^{R} w^r \cdot w^{r'} \beta(r')^{-2} = 1$$: Normalization condition

($\beta(r)$ ~ Scalar field)

Examples : Homogenous spheres

(More eg. in Taigen's poster)

- Homogeneous S^2

$a = (l, m)$ $\quad$ l, m : integers, $|m| \leq l \leq L$ $\quad$ L : Cut-off

$$P_{(l_1 m_1)\,(l_2 m_2)\,(l_3 m_3)} = \int_{S^2} d\Omega \; \tilde{Y}_{l_1 m_1}(\Omega)\, \tilde{Y}_{l_2 m_2}(\Omega)\, \tilde{Y}_{l_3 m_3}(\Omega)$$

$$\tilde{Y}_{lm}(\Omega) = \begin{cases} \frac{1}{\sqrt{2}} \left(Y_{lm}(\Omega) + Y^*_{lm}(\Omega) \right) e^{-l^2/L^2} & (m > 0) \\ Y_{l0}(\Omega)\, e^{-l^2/L^2} & (m = 0) \\ \frac{1}{\sqrt{2} i} \left(Y_{lm}(\Omega) - Y^*_{lm}(\Omega) \right) e^{-l^2/L^2} & (m > 0) \end{cases}$$

$Y_{l,m}(\Omega)$: Spherical harmonics with $\Omega = (\theta, \phi)$

e^{-l^2/L^2} : Regulator to smooth the cutoff. Necessary for locality.

- Homogeneous S^n

$$a = (l_1 \dots l_n) \qquad |l_1| \leq l_2 \leq \cdots \leq l_n \leq L$$

$$P_{(l_1 \dots l_n)\,(l'_1 \dots l'_n)\,(l''_1 \dots l''_n)} = \int_{S^n} d\Omega\, \tilde{Y}_{l_1 \dots l_n}(\Omega)\, \tilde{Y}_{l'_1 \dots l'_n}(\Omega)\, \tilde{Y}_{l''_1 \dots l''_n}(\Omega)$$

$Y_{l_1 l_2 \dots l_n}(\Omega)$: n-dimensional spherical harmonics

$\tilde{Y}_{l_1 \dots l_n}(\Omega)$: Real combinations of $Y_{l_1 l_2 \dots l_n}(\Omega)$ with a regulator.

The tensor-rank decomposition was done numerically:

$$\min_{v_a^r} \left| P_{abc} - \sum_{r=1}^{R} v_a^r v_b^r v_c^r \right|^2$$

R is chosen so that the remaining error be a few percents or so.

(We used our own C++ program.)

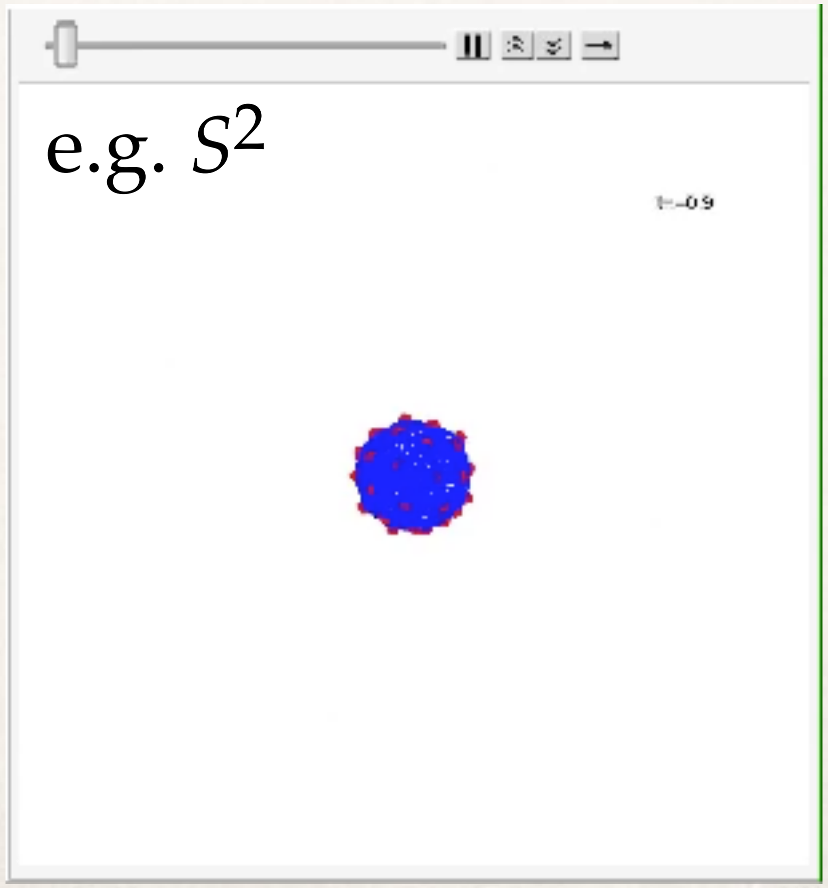
We numerically solved the classical e.o.m. of CTM, and performed tensor-rank decompositions, etc, of $P_{abc}(t)$ at some representative times:

Initial conditions: Homogenous spheres S^1, S^2, S^3.

Evolution parameters: Homogeneous N_a, $N_{ab} = 0$.

$$\text{e.o.m.:} \quad \frac{d}{dt}P_{abc} = \{P_{abc}, \mathscr{H}\} = -N_d P_{de(a}P_{bc)e} + N_{d(a}P_{bc)}$$

$$P_{abc}|_{t=0} = P_{abc} \text{ of homogeneous } S^{1,2,3}$$



The topology does not change.

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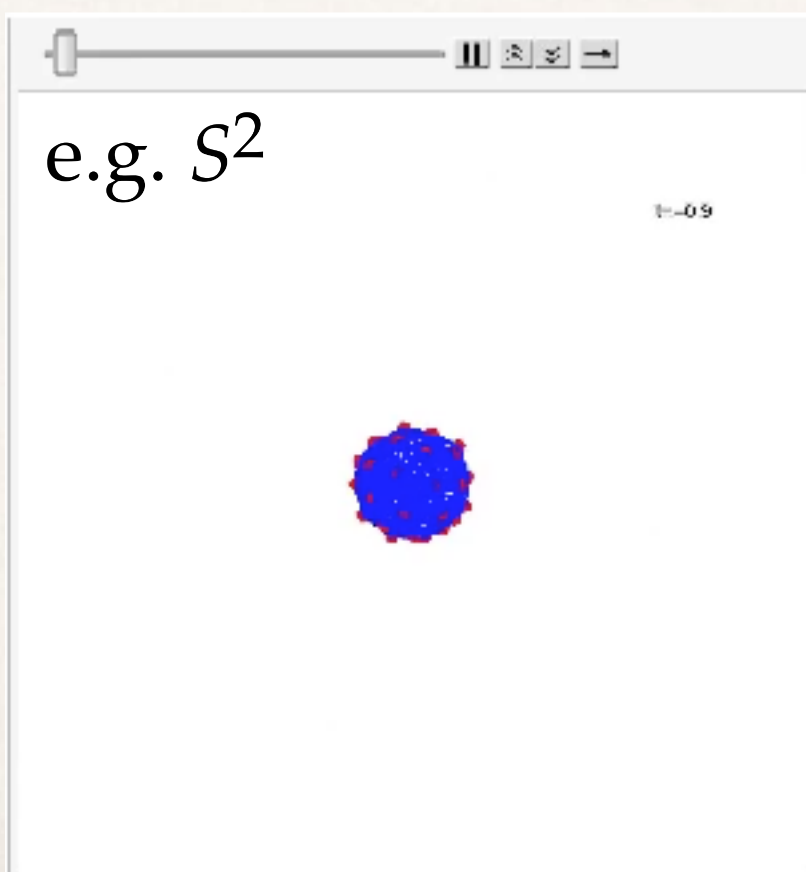
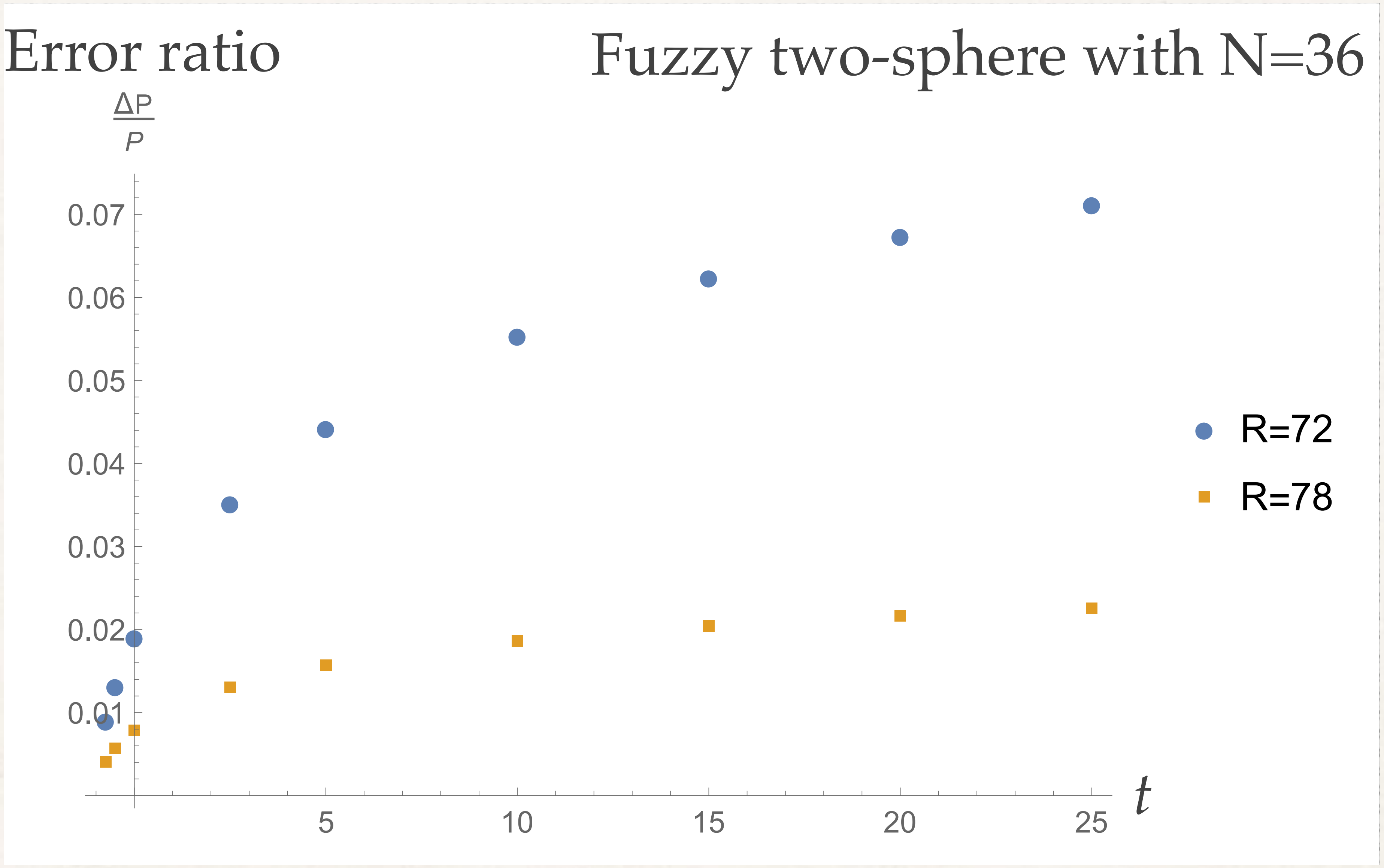


The topology does not change.

(i) The time evolution increases “complexity” of the spaces.

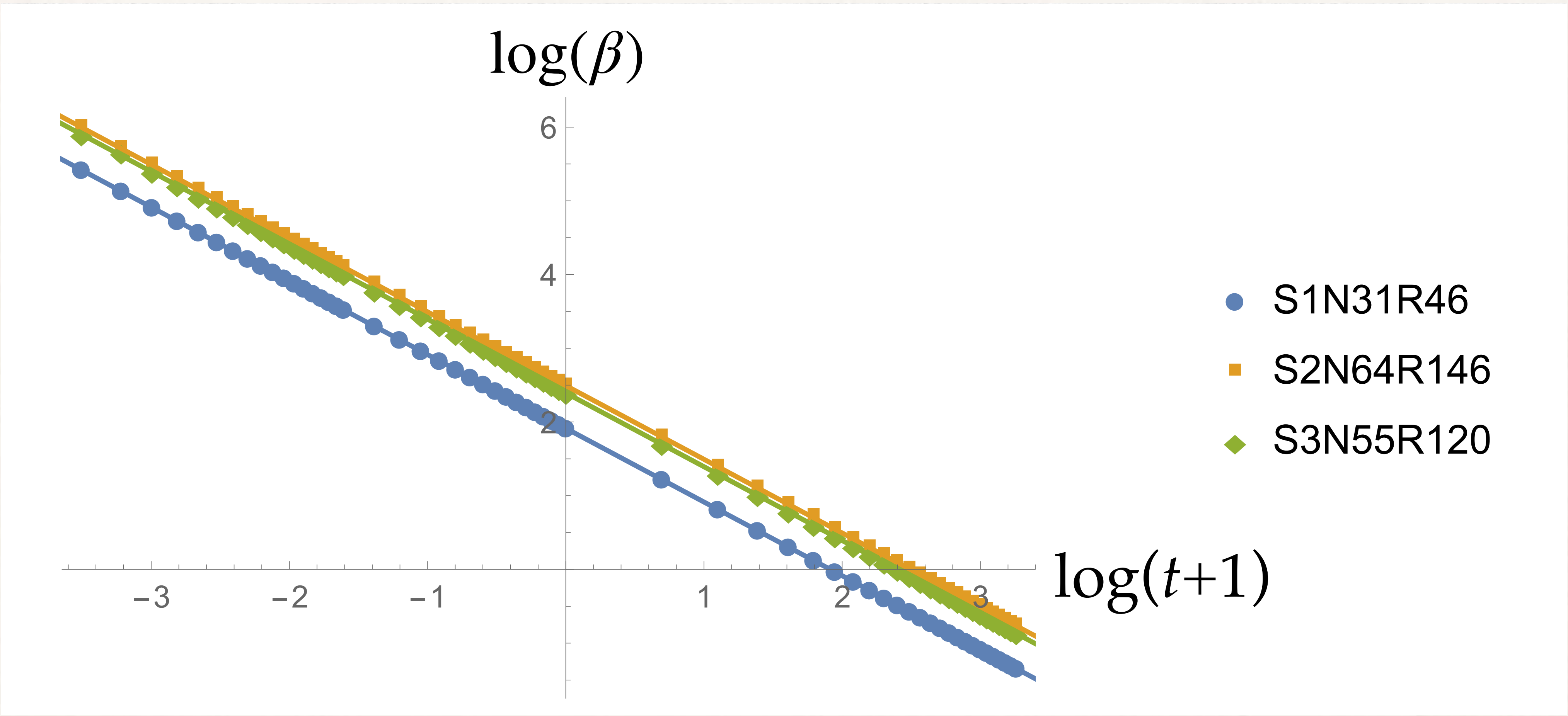
The remaining error of the tensor-rank decomposition increases in time for fixed R. In other words, one has to increase the number of points to describe spaces with fixed correctness, as time passes.



$$\Delta P^2 \equiv \left(P_{abc} - \sum_{i=1}^{R} v_a^i v_b^i v_c^i\right)^2$$

(ii) The time evolutions agree with that of the general relativistic system obtained previously in the formal continuum limit.

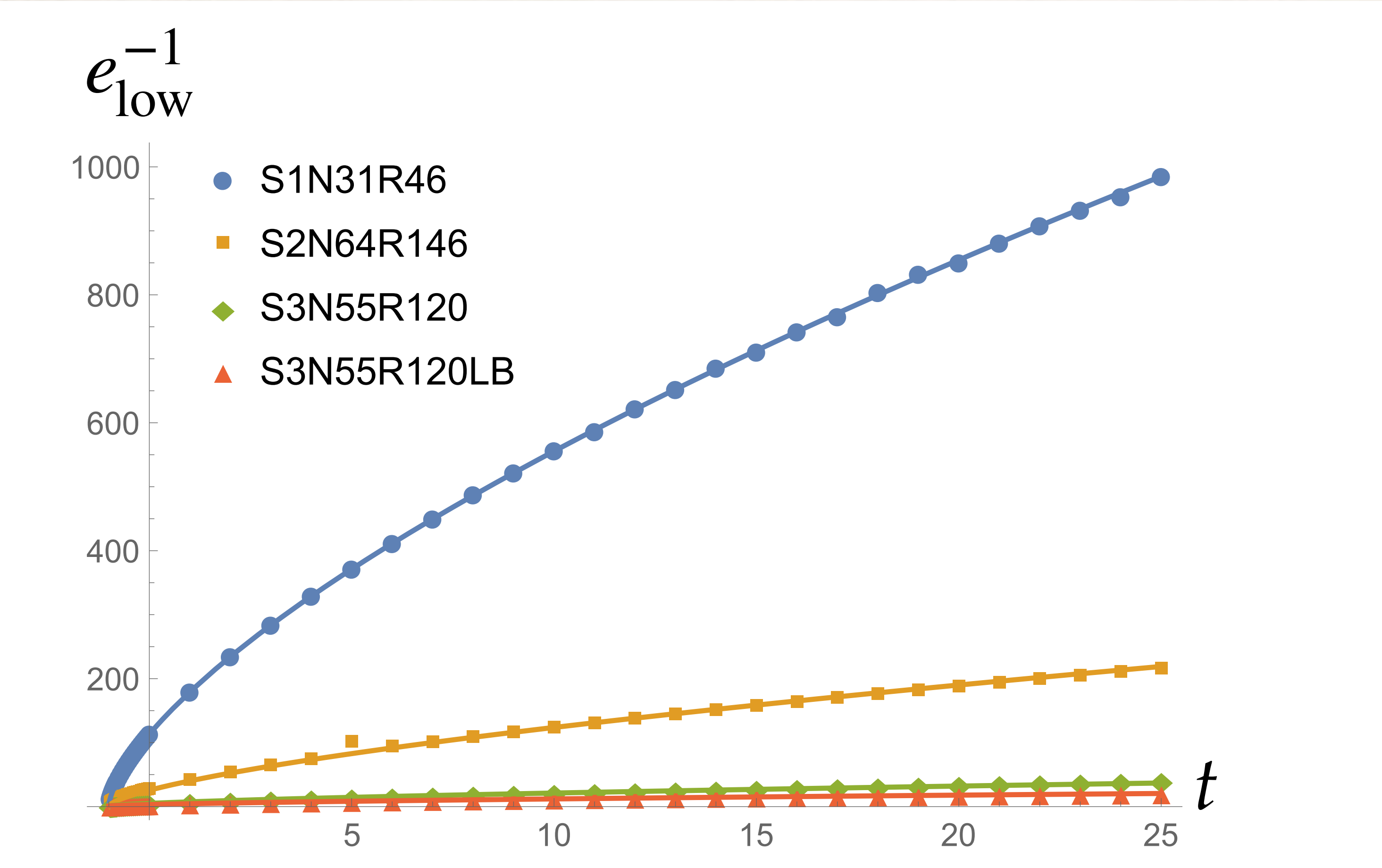
Time evolution of the scalar field



Dots: From P_{abc} at representative times in canonical tensor model

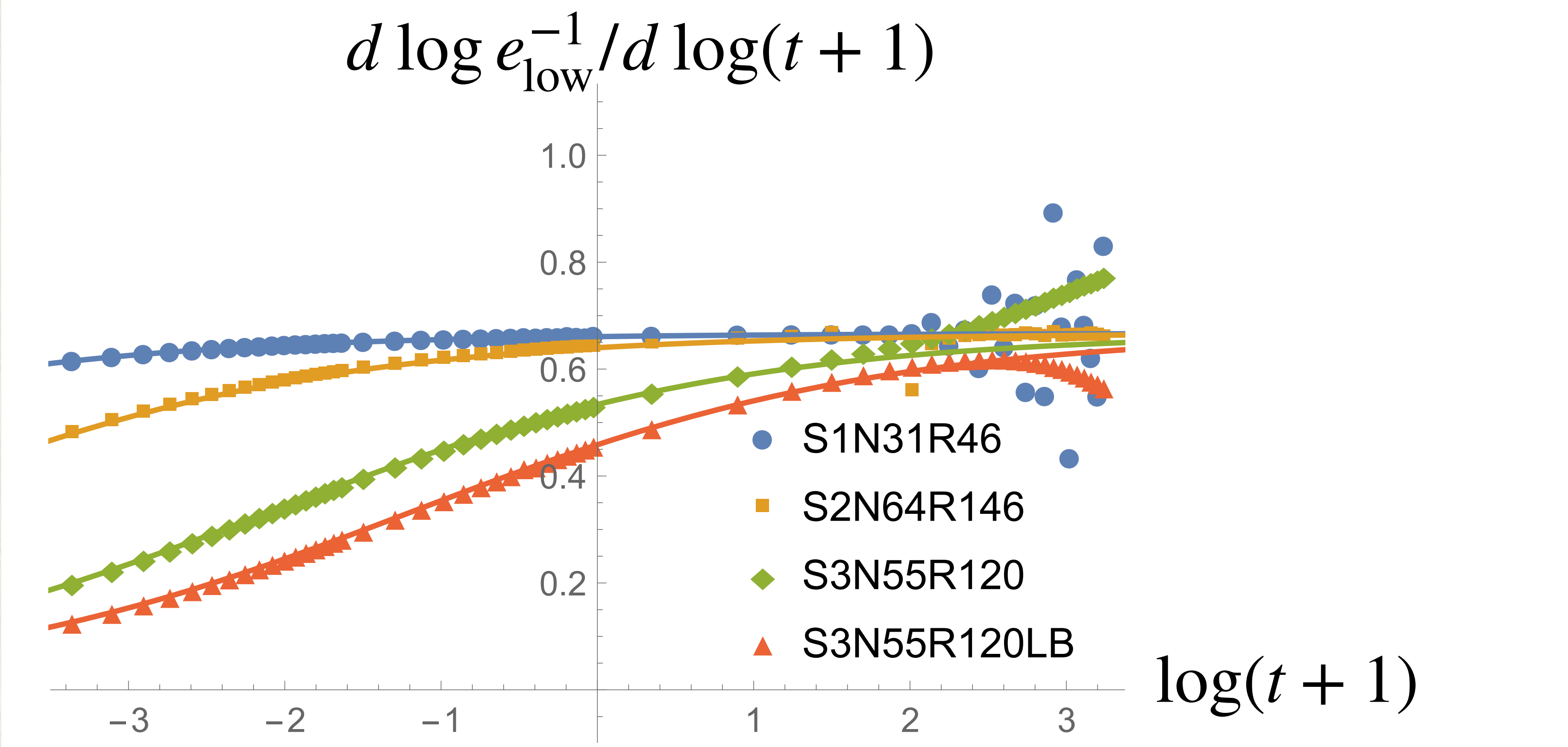
Lines: General relativistic system

Time evolutions of the sizes of the spaces



$e_{\rm low}$: the lowest eigenvalue of -K

$\propto$ (the size of a space)$^{-2}$

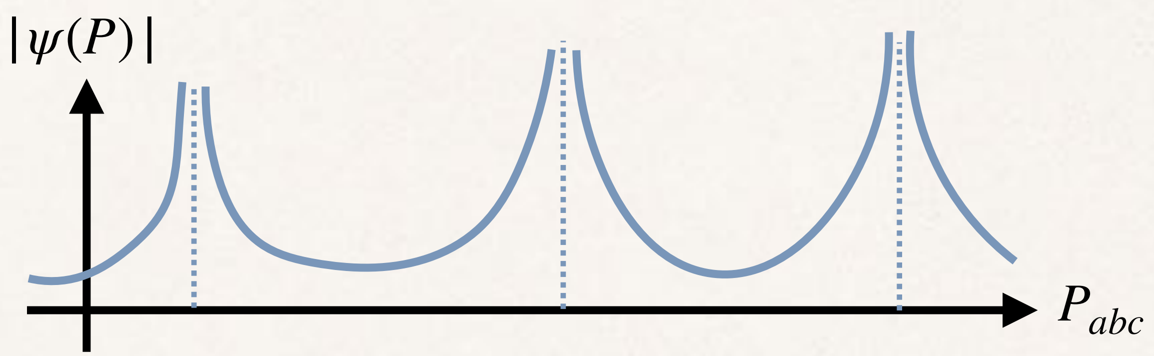


Summary

- The techniques of data analysis, namely, tensor-rank decomposition, persistent homology, and Laplace operator, gives a spacetime (topological and geometric) interpretation of the canonical tensor model.
- The real symmetric three-indexed tensor can in principle describe any smooth spaces with any topologies and dimensions. (Taigen's poster)
- The classical time evolutions of the homogeneous spheres agrees with that of the general relativistic system.

Future problems

- Extend analysis to larger systems. Topology changes, singularities, …
- What are black hole, horizon, singularity, general relativity, energy, momentum, black hole temperature/entropy,…, in data analysis ?
- What are the spacetime interpretation of the configurations at peaks?



- Which is a space, Q_{abc} or P_{abc} ? Actually, in stationary phase approx.,

$$\hat{Q}_{abc}\psi \sim \int_H dh\, \phi_a(h)\, \phi_b(h)\, \phi_c(h) + \int_{H'} + \cdots$$

Group manifold H appears in a continuous tensor-rank decomposition.